J. Fluid Mech. (1976), *vol.* 77, *part* 3, *pp.* 433–472
Printed in Great Britain

On over-reflexion

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(Received 8 August 1975 and in revised form 3 March 1976)

Reflexion coefficients greater than unity have now been predicted for a variety of different systems involving waves propagating towards a shear layer, but almost invariably only in regions of parameter space for which the layer exhibits Kelvin–Helmholtz instability. This paper contains a study of two examples in which, for appropriate parameter values, there are no such instabilities to obscure (or even prevent) the 'over-reflexion' of an incident wave, namely (*a*) hydromagnetic internal gravity waves meeting a vortex–current sheet in a stratified fluid and (*b*) magneto-acoustic waves meeting a vortex sheet in a compressible fluid. In the former case the energetic aspects of over-reflexion are examined in detail, thus displaying the way in which the excess reflected energy is extracted from the mean motion and the sense in which the transmitted wave may be viewed, by analogy with certain concepts employed in plasma physics, as a carrier of so-called 'negative energy'.

1. Introduction and summary

It has been known for nearly twenty years that linear theory occasionally predicts, for waves of one kind or another incident upon a shear layer, a reflexion coefficient greater than unity, so that more energy is reflected back towards the source than was originally emitted. This 'over-reflexion' was first encountered in studies by Miles (1957) and Ribner (1957) of the transmission and reflexion of sound waves at a vortex sheet $z = 0$ separating two regions of constant horizontal velocity U_1 and U_2. This acoustic problem was subsequently extended by Fejer (1963) to include hydromagnetic effects and by McKenzie (1972) to include effects due to buoyancy. The over-reflexion of internal gravity waves by a finite layer of constant shear, as opposed to a vortex sheet, separating two uniform streams of incompressible fluid has been investigated numerically by Jones (1968) and Breeding (1971) and analytically by Eltayeb & McKenzie (1975). The over-reflexion of Rossby waves propagating through a hyperbolic-tangent zonal shear flow on a beta-plane has been examined numerically by Dickinson & Clare (1973) and Geisler & Dickinson (1974).

Precisely what significance can be attached to over-reflexion has however been somewhat obscured by the fact that it appears to have been predicted only in circumstances for which the vortex sheet/shear layer is unstable. When internal gravity waves meet a finite layer of constant shear, for example, over-reflexion cannot occur unless the Richardson number Ri is less than $\frac{1}{4}$. Although it is well

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known that $Ri < \frac{1}{4}$ does not automatically imply instability (while $Ri > \frac{1}{4}$ certainly implies stability) of a stratified shear flow (see Drazin & Howard 1966; Hazel 1972; Howard & Maslowe 1973), Jones (see his 'model III') and Breeding have shown that it does in this particular case. When acoustic waves meet a vortex sheet, on the other hand, a purely two-dimensional analysis (i.e. 'crests' normal to both the z axis and the direction of streaming) predicts that at sufficiently high Mach number over-reflexion can occur without instability. Fejer & Miles (1963) have, however, shown that the system is always unstable to suitably *three*-dimensional disturbances at any Mach number.

The presence of instabilities *per se* by no means, of course, renders insignificant any prediction of over-reflexion. Whether it will be able to occur and be clearly discernible in spite of them will presumably then depend on the magnitudes of their growth rates and on the rate at which they modify the basic flow, but this aspect of the problem is not usually investigated. One notable exception is the numerical study of Rossby-wave over-reflexion by Geisler & Dickinson (1974). Over-reflexion is found to occur only when the mean flow is such that the potential vorticity gradient $\beta - d^2U/dy^2$ (y here denoting northward distance from some fixed latitude and β denoting the gradient of the Coriolis parameter) changes sign at some latitude. The simple and well-known sufficient condition for stability due to Kuo (1949) is then violated, and by decoupling the waves and mean flow in the numerical model Geisler & Dickinson indeed revealed the presence of instabilities. Nevertheless throughout most of their investigation the waves and mean flow were coupled and the former did *not* then experience any sustained growth, owing (presumably) to the time required for significant growth of the instabilities being longer than that required for the over-reflexion of the incident wave to cause significant changes in the mean flow (by extracting kinetic energy from it). It is important to note that only one zonal wavenumber was carried in the numerical calculations, but it was nevertheless argued quite convincingly, from the results of Dickinson & Clare (1973), that normal modes of other wavenumbers were also unlikely to have large enough growth rates to prevent the over-reflexion process.

We shall show in this paper that it is possible to find systems in which, for suitable values of the relevant parameters, there are *no* such unstable modes which could mask or even prevent the over-reflexion of an incident wave, at least on the basis of a linear, diffusionless theory in which the shear layer is modelled by a vortex sheet.†

The central problem investigated in this paper is the reflexion of hydro-

† Each of these three qualifications covers a serious possibility requiring investigation. We can comment here only very briefly on each in lieu of the very substantial further analysis that will be required. It is possible, first, that the systems are subcritically unstable to finite-amplitude disturbances. Second, it is possible that even for very small values of the diffusion coefficients there are instabilities analogous to the 'resistive instabilities' of, for example, Baldwin & Roberts (1972), even when the non-dissipative stability criterion is satisfied. Finally, it is possible that by using a vortex-sheet model we exclude a class of unstable modes which would exist for a layer of finite thickness, again even if (1.1) or its equivalent were satisfied, as has been found by Blumen, Drazin & Billings (1975) in the purely acoustic problem, although it seems reasonable on the basis of their work to hope that (if indeed there were such modes) the growth rates would be very small if the shear-layer thickness were also small.

magnetic internal gravity waves propagating in an incompressible fluid towards a vortex–current sheet. The density of the fluid is taken as a gently and continuously decreasing function of height, while the magnetic field, which is taken parallel to the mean flow and is conveniently measured by its associated Alfvén speed V [see equation (2.9)], takes different (constant) values V_1 and V_2 on the two sides of the sheet. The magnetic field has two effects: it stabilizes the system (subject to the above qualifications) if

$$V_1^2 + V_2^2 > \tfrac{1}{2}(U_2 - U_1)^2 \tag{1.1}$$

but at the same time tends to suppress over-reflexion, so that this is only possible if

$$|V_1| + |V_2| < |U_2 - U_1|. \tag{1.2}$$

Provided there *is* a current sheet, i.e. $V_1 \neq V_2$, these inequalities leave a (comparatively small) stable region of parameter space in which waves *of suitable horizontal phase speed* will be over-reflected, as shown in figure 1. The relevant analysis is given in §§ 2 and 3.

Another such example is furnished by magneto-acoustic waves propagating towards a vortex sheet in a compressible fluid free from the action of gravity. Both the reflexion problem and the associated stability problem have been extensively studied (Fejer 1963, 1964; Sen 1965; Southwood 1968; McKenzie 1970; Ong & Roderick 1972; Duhau & Gratton 1973) but the results are extremely complicated and in the case of the stability problem are available only for a few limiting cases. As a result we have so far been able to delineate (appendix C) stable over-reflecting regimes only when there is simply a discontinuity in velocity, the densities, sound speeds and magnetic fields (which we again take parallel to the streaming motion) being the same on both sides of the sheet. The results are displayed in figure 2, where

$$A = a/|V|, \quad W = |U_2 - U_1|/2|V| \tag{1.3}$$

and a is the speed of sound.

It is probably worth pointing out at this early stage that the above statements are by no means in contradiction with what is intuitively quite clear, namely that if the over-reflected wave is subsequently returned (without too much, if any, loss of amplitude) to the vortex–current sheet, by reflexion at either a solid boundary or another vortex–current sheet at some other level, it will inevitably keep increasing in amplitude by multiple over-reflexion (see § 5 and appendix B). It has no such opportunity to return to the *single* vortex–current sheet in the above *unbounded* systems.

In this paper we try also to shed some light on the relationship between over-reflexion and critical-layer absorption. It is tempting to view the two phenomena as simply opposite ends of the same spectrum, critical-layer absorption occurring if the velocity gradient in the shear layer is sufficiently weak (i.e. $Ri > \frac{1}{4}$ in the pure internal gravity wave problem; see Booker & Bretherton 1967) and over-reflexion occurring only if the gradient is sufficiently strong (i.e. $Ri < \frac{1}{4}$), but there do appear to be some important differences (see § 5). It is known, for example, that there are circumstances in which critical-layer absorption can

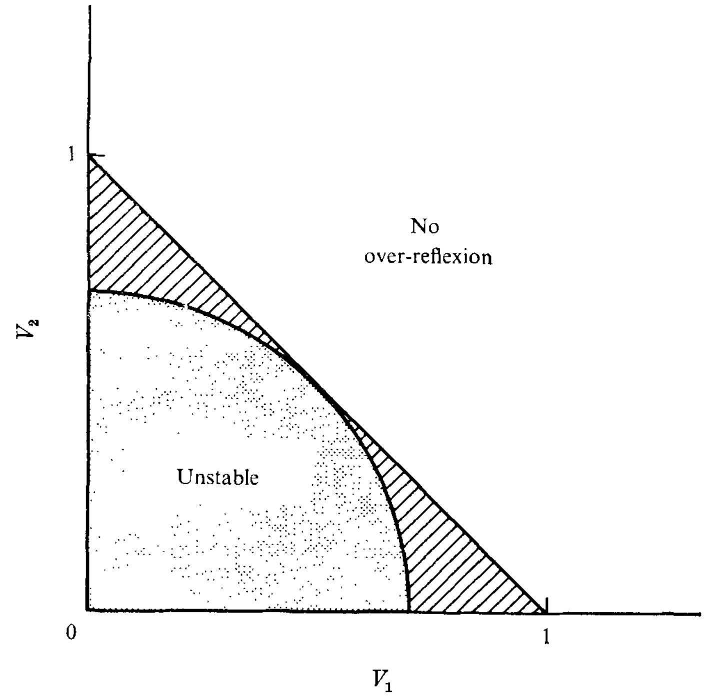


FIGURE 1. Illustrating the stable portion of the V_1, V_2 plane in which hydromagnetic internal gravity waves of suitable horizontal phase speed will be over-reflected (hatched). The Alfvén speeds have here been made dimensionless with respect to the jump $|U_2 - U_1|$ in horizontal velocity across the sheet.

occur even in the absence of a basic shear flow, or indeed any basic flow at all (Acheson 1972, 1973), but it seems highly unlikely that the same is true of over-reflexion.

The third objective of this paper is to reveal quite clearly, at least in one special case (i.e. depth of shear layer $d \ll$ vertical wavelength λ, in which case the simple model problem of § 3 should provide an adequate description of events; see § 5), the energetic aspects of the over-reflexion mechanism. In order to do this we have extended some results from the rapidly growing literature on the interaction between internal gravity waves and mean shear flows to the hydromagnetic case (§§ 2, 4 and appendix A), and these should remain of value when more realistic models than that in § 3 are considered. As far as the author can tell, these energetic aspects and certain careful distinctions which have to be made in their elucidation, collected together below for the sake of clarity, should be quite typical (provided $d \ll \lambda$) of any other over-reflecting system.

When hydromagnetic internal gravity waves propagate through a fluid *at rest* there is at any level z a mean upward flux of energy

$$F = \overline{pw}, \tag{1.4}$$

where p is the total (i.e. including magnetic) perturbation pressure, w is the vertical velocity and an overbar denotes a horizontal average. If the basic state

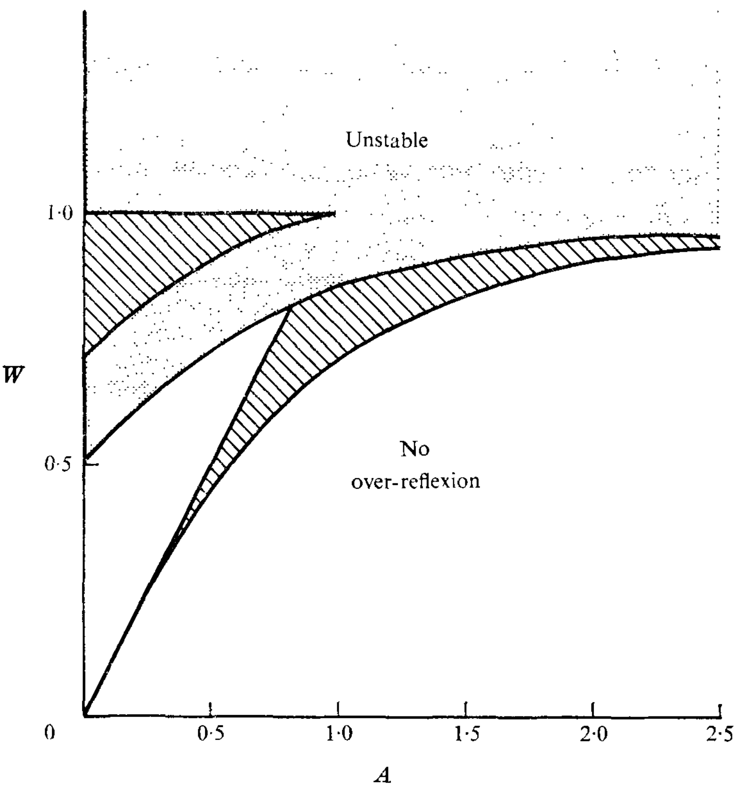


FIGURE 2. Illustrating, when the densities, sound speeds and magnetic fields on the two sides of the vortex sheet are equal, the stable portion of the A, W plane in which magneto-acoustic waves of suitable horizontal phase speed will be over-reflected (hatched).

is one of horizontal magnetic field $B(z)$ *and horizontal velocity* $U(z)$, however, it is important in evaluating the upward flux of energy that one either (*a*) takes account of the fact that there is then, in addition to (1.4), indirect vertical advection of energy across surfaces of constant height due to the waves [see appendix A especially equation (A 4)] or (*b*) computes the mean rate of working of pressure forces alone but on *the* (*sinusoidally*) *distorted material surface* composed of fluid particles that were at the height z in the equilibrium state [see (2.20) and (A 5)]. Either way, terms additional to (1.4) arise in the expression for the *net* mean upward energy flux, which is given by

$$\mathscr{F} = \overline{pw} + U(\rho_0 \overline{uw} - \mu^{-1}\overline{b_x b_z}) \tag{1.5}$$

[see (2.19) and (A 7)]. The extra terms are nevertheless simply related to $\overline{pw}$ [see equation (2.26)] in such a way that

$$\mathscr{F} = c\overline{pw}/(c-U), \tag{1.6}$$

where c is the horizontal phase speed of the wave relative to the fixed co-ordinate system. The quantity F, which, following previous authors, we term the 'wave energy flux', thus represents the net mean upward energy flux only as measured by an observer moving horizontally with the speed of the local mean flow $U(z)$. This crucial distinction between F and $\mathscr{F}$, though emphasized in the non-hydromagnetic case by Hines & Reddy (1967) and Lindzen (1973), appears to have been quite often overlooked in the recent literature, with occasionally erroneous results. One such instance, to which the present paper is especially pertinent, is discussed in §5.

The considerations so far have been valid even if the basic flow $U(z)$ and magnetic field vary with height, and both F and $\mathscr{F}$ are then subject to certain important constraints (see §2) of value when considering over-reflexion at a finite shear layer rather than a vortex sheet (see §5). In the case when U and B are constant (or vary only slightly over vertical distances of the order of a wavelength) however, a vertical wavenumber and hence a vertical component of group velocity w_g can be well defined (at least locally) and then

$$F = Ew_g. \tag{1.7}$$

Here E is the 'mean wave energy density', an intrinsically positive quantity, being calculated in the usual way as the horizontally averaged sum of squares of appropriate *perturbation* quantities [see (2.34)]. Clearly, by (1.6) the *net* mean upward flux of energy in these circumstances is not (1.7) but

$$\mathscr{F} = \mathscr{E}w_g, \tag{1.8}$$

as if a 'net' energy density

$$\mathscr{E} = cE/(c-U) \tag{1.9}$$

were being carried upwards at the group velocity.

If we now consider the steady-state over-reflexion (§3) in its simplest form, namely when the lower fluid is at rest ($U_1 = 0$, $U_2 = U > 0$), the incident wave (from below) will have a positive $\mathscr{F}$ ($= F$) and w_g, while those associated with the reflected wave will be negative. In over-reflecting circumstances the separate $\mathscr{F}$'s may be simply added (§3) to give a net downward energy flux in the lower region. Further, the appropriate interface conditions, i.e. continuity of vertical displacement and perturbation pressure p, are easily seen to imply that $\mathscr{F}$ (but not F) will be continuous across the vortex–current sheet, which acts as neither a source nor a sink of energy. Thus the values of $\mathscr{F}$ in both the upper and the lower region must be the same and *downward*. Since the radiation condition on the transmitted wave in the upper region implies that w_g and F are positive there (see §§3 and 4), it then follows from (1.8) and (1.9) that $0 < c < U$, so that over-reflexion can occur only if the horizontal phase speed of the wave lies between the two fluid speeds. It also follows that $\mathscr{E}$ is *negative* in the upper region, so that the transmitted wave might be thought of as carrying 'negative energy' $\mathscr{E}$ upwards at the group velocity.

This terminology, borrowed (e.g. McKenzie 1970, 1972) from the theory of microwave tubes and space-charge waves on electron plasma beams† (see, for example, Krall & Trivelpiece 1973, pp. 136–143; Sturrock 1960, 1961, 1962;

† A space-charge wave is said to have 'positive' or 'negative' energy according as its phase speed is greater or less than the average electron drift velocity. This may be compared with (1.9), where $\mathscr{E}$ is positive or negative according as c is greater or less than U.

Pierce 1974, pp. 102–113; Kadomstev, Mikhailovskii & Timofeev 1965), recognizes that as the wave propagates through the upper region it must somehow cause a net *reduction*, rather than enhancement, of the total energy there, and this reduction is in fact precisely that needed to account for the surplus energy heading back towards the source as a result of the over-reflexion. But a calculation of the total energy in any region requires being able to keep track of any second-order effects due to the waves, i.e. any changes in the mean flow or magnetic field. These in turn cannot be determined from consideration of the final steady-state wave system alone, and require consideration of how that wave system was established. Such time-dependent considerations also seem desirable if, with the net energy flux $\mathscr{F}$ being downward throughout, one is to be fully confident that the radiation condition has been correctly applied to the transmitted wave.

Accordingly in §4 the very gradual establishment of a constant-amplitude hydromagnetic wave train in a moving stratified fluid is investigated by a multiple-scale method, from which it becomes clear precisely how the transmitted wave steadily diminishes the total energy in the upper region.† As the effective 'front' reaches a given level, it introduces wave energy E that was not there before, but its precursor has already gradually set up a change $\overline{u}$ in the mean horizontal flow (but not magnetic field) at that level given by

$$\rho_0 \overline{u} = E/(c-U), \tag{1.10}$$

which, when $c < U$, means a (time-independent thereafter) decrease in the mean horizontal flow and hence in the kinetic energy associated with it. The change in total energy (density) is therefore evidently

$$E + \rho_0 U \overline{u}, \tag{1.11}$$

i.e. the 'wave energy' plus U times the momentum change induced by the wave, and is precisely equal to $\mathscr{E}$. As the transmitted wave propagates upwards it simply steadily decreases the kinetic energy of the mean horizontal flow [in more and more of the upper region and by a constant amount (1.10)] *at a greater rate* than it brings in 'wave energy' E to the upper region. The process is illustrated very schematically in figure 3 for the transient case in which a slowly modulated incident wave train, with a beginning and an end, meets the vortex–current sheet. It is the alterations in the mean flow that are displayed in figure 3, and these show clearly what range of heights the various wave trains occupy at each stage of the process. The transmitted wave may or may not be larger than the incident one [see (3.6)] but as long as $c < U$ the defect in mean flow energy in the upper region exceeds the transmitted 'wave' energy, and over-reflexion occurs.

Whether, therefore, the wave is to be regarded, when $c < U$, as a 'negative energy' one or not depends simply on whether one wishes to count the concomitant change in kinetic energy of the mean flow as part of the wave energy or to count it separately, so that ultimately, as Bretherton (1969*a*) remarks, whether $\mathscr{E}$ or E should be regarded as 'the' energy associated with the wave is largely

† Similar results have been obtained independently, for a special case of the problem considered here, by McIntyre & Weissman (1976).

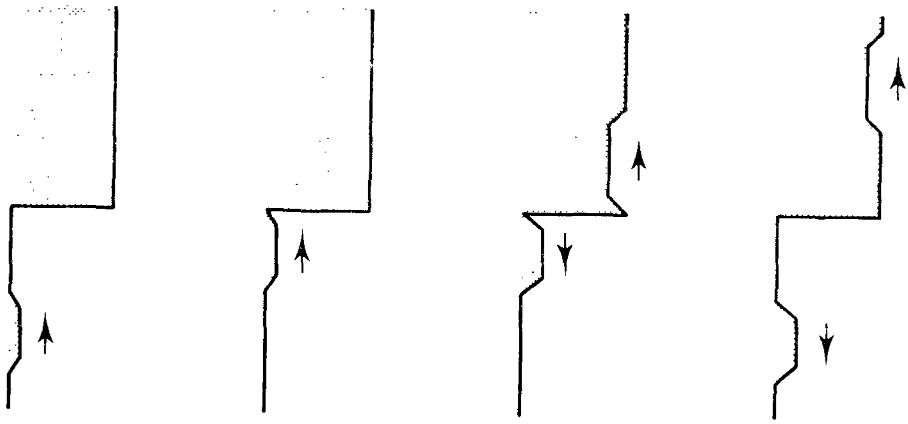


FIGURE 3. Schematic illustration of the variation with time of the mean horizontal flow during a transient over-reflexion. The lower fluid is at rest.

a matter of personal preference. We consistently take the latter viewpoint in this paper (noting, in passing, that there are good reasons for doing so when dissipative effects are included – see §5 – and in wave problems of a more general type in which, for example, the wave amplitude is modulated not only in the z direction but in the x direction also). Nevertheless the physics of the problem are elucidated here by reference to a fixed co-ordinate frame, so the quantities $\mathscr{E}$ and $\mathscr{F}$ are crucial, and we finally note the close relationship between them and the wave *action* (e.g. Bretherton & Garrett 1968; Bretherton 1971). This emerges clearly from §4, where the time evolution of a hydromagnetic–gravity wave train whose amplitude varies slowly with height is analysed when the mean flow and magnetic field also vary slowly with height. The wave-action density is defined as

$$\mathscr{A} \equiv E/(\omega - Uk), \tag{1.12}$$

where ω is the frequency of the wave relative to the fixed co-ordinate frame and k is the horizontal wavenumber. We see from (1.8) and (1.9) that

$$\mathscr{E} = \omega\mathscr{A}, \quad \mathscr{F} = \omega w_g \mathscr{A}, \tag{1.13}$$

so that in the present system the wave action is simply a constant multiple of what we have termed above the 'net' change in energy due to the waves, $\mathscr{E}$. The amplitude evolution of the wave train is shown in §4 to be governed by the conservation of wave action (as also follows from Bretherton & Garrett's general argument):

$$\frac{\partial \mathscr{A}}{\partial t} + \frac{\partial}{\partial z}(w_g \mathscr{A}) = 0. \tag{1.14}$$

Using a number of the results quoted above, including (1.13), this can be cast into the following form involving E and F:

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial z} + \frac{dU}{dz}(\rho_0 \overline{uw} - \mu^{-1}\overline{b_x b_z}) = 0. \tag{1.15}$$

This equation involving the 'wave energy' E, 'wave energy flux' F and a radiation–stress term (Longuet-Higgins & Stewart 1964; Whitham 1962; Bretherton 1971, pp. 96–97) can also be derived directly from the linearized perturbation equations, or (appendix A) using the conservation of total energy. On the other hand, using (1.13) equation (1.14) can also be written in the form

$$\partial \mathscr{E}/\partial t + \partial \mathscr{F}/\partial z = 0, \tag{1.16}$$

making the consistency between conservation of wave action and *total* energy in the present system especially transparent.

2. Hydromagnetic waves of steady amplitude in a stratified shear flow

When all diffusive processes can be neglected the basic hydromagnetic equations governing the motion of an incompressible fluid are

$$\rho(\partial \mathbf{u}/\partial t + \mathbf{u}.\nabla \mathbf{u}) = -\nabla p + \mu^{-1}\mathbf{B}.\nabla \mathbf{B} + \rho \mathbf{g}, \tag{2.1}$$

$$\partial \mathbf{B}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}), \tag{2.2}$$

$$\nabla . \mathbf{u} = 0, \quad \nabla . \mathbf{B} = 0, \qquad (2.3), (2.4)$$

$$\partial \rho/\partial t + \mathbf{u}.\nabla \rho = 0 \tag{2.5}$$

(see, for example, Chandrasekhar 1961, chap. 4). Here ρ denotes the fluid density, $\mathbf{u}$ the Eulerian velocity vector, t time, μ the magnetic permeability, $\mathbf{B}$ the magnetic field, $\mathbf{g}$ the acceleration due to gravity and $p = p_F + \frac{1}{2}\mu^{-1}\mathbf{B}^2$ is the 'total' pressure, including both the fluid pressure p_F and the magnetic pressure $\frac{1}{2}\mu^{-1}\mathbf{B}^2$.

Referring all quantities to a set of rectangular Cartesian co-ordinates (x, y, z) the basic equilibrium state

$$\mathbf{u}_0 = \{U(z), 0, 0\}, \quad \mathbf{B}_0 = \{B(z), 0, 0\} \tag{2.6}$$

is an exact solution of (2.1)–(2.5) provided that the density varies with height only, in which case magnetohydrostatic balance holds:

$$dp_0/dz = -\rho_0 g, \tag{2.7}$$

where $\mathbf{g} = (0, 0, -g)$.

If we now consider small-amplitude perturbations $\mathbf{u}$, $\mathbf{b}$, ρ and p (in velocity, magnetic field, density and total pressure respectively) to this system the linearized forms of (2.1)–(2.5) admit two-dimensional solutions in which all perturbation quantities ψ may be written as

$$\psi = \mathscr{R}[\hat{\psi}(z) \exp i(kx - \omega t)]. \tag{2.8}$$

After making the Boussinesq approximation, in which the basic density gradient is supposed so weak that ρ_0 may be regarded as taking some constant mean value everywhere in (2.1) except in the buoyancy term, we eliminate all perturbation quantities in favour of the vertical velocity $\hat{w}$. Thus, defining the local Alfvén speed

$$V(z) \equiv B(z)/(\mu \rho_0)^{\frac{1}{2}}, \tag{2.9}$$

the buoyancy (Brunt–Väisälä) frequency (assumed constant and real, so that the fluid is stably stratified)

$$N \equiv \left(\frac{-g}{\rho_0}\frac{d\rho_0}{dz}\right)^{\frac{1}{2}} \tag{2.10}$$

and the function

$$P(z) \equiv V^2/(c-U)^2 - 1 \tag{2.11}$$

(where $c = \omega/k$, the horizontal phase speed), we find

$$P\hat{w}'' + P'\hat{w}' + \left[\frac{(PU''+P'U')}{(c-U)} - Pk^2 - \frac{N^2}{(c-U)^2}\right]\hat{w} = 0, \tag{2.12}$$

where the primes denote differentiation with respect to z. The other perturbation quantities are related to $\hat{w}$ as follows:

$$\hat{u} = i\hat{w}'/k, \quad \hat{b}_z = B\hat{w}/(U-c), \qquad (2.13), (2.14)$$

$$k\hat{b}_x = \frac{-i}{c-U}\left(B\hat{w}' + \frac{BU'\hat{w}}{c-U} + B'\hat{w}\right), \tag{2.15}$$

$$k\hat{p} = -i\rho_0 P\{(c-U)\hat{w}' + U'\hat{w}\}, \tag{2.16}$$

$$k(c-U)\hat{\rho} = -i\rho_0'\hat{w}. \tag{2.17}$$

Energy flux

We have already made a clear distinction in §1 between the *wave energy flux*, defined as

$$F = \overline{pw}, \tag{2.18}$$

and the *net* energy flux at any given height z_0,

$$\mathscr{F} = \overline{pw} + U(\rho_0\overline{uw} - \mu^{-1}\overline{b_x b_z}). \tag{2.19}$$

The origin of the additional terms in (2.19) through vertical advection of energy across surfaces of constant height is demonstrated in appendix A [see especially equation (A 4)]. A simple alternative way of seeing how they arise is to calculate the rate of working of the perturbation pressure forces alone, but normal to the sheet $\mathscr{S}$ composed of fluid particles which *were* at z_0 in the absence of the waves. If the equation of this sheet is $z - z_0 = \eta(x-ct)$ and s denotes arc length along it, the mean rate of working by perturbation pressure forces of the fluid below on that above is measured by

$$\int p(\mathbf{U}+\mathbf{u}).\mathbf{n}\,ds, \tag{2.20}$$

where

$$\mathbf{n} = (-\partial\eta/\partial x, 0, 1)[1+(\partial\eta/\partial x)^2]^{-\frac{1}{2}} \tag{2.21}$$

denotes the unit normal to $\mathscr{S}$, $\mathbf{U}$ denotes the undisturbed flow $(U, 0, 0)$ and the integral (2.20) is taken over a wavelength. Thus (2.20) may be written as

$$\begin{aligned}\int p\left(w - U\frac{\partial\eta}{\partial x}\right)dx &= \int pw\,dx + U\int\eta\frac{\partial p}{\partial x}\,dx \\ &= \int pw\,dx + U\int\eta\left[-\rho_0\left(\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + wU'\right) + \frac{1}{\mu}\left(B\frac{\partial b_x}{\partial x} + b_z B'\right)\right]dx, \end{aligned} \tag{2.22}$$

where the linearized x component of the momentum equation (2.1) has been used in the last line. Now

$$w = \partial\eta/\partial t + U\,\partial\eta/\partial x, \quad b_z = B\,\partial\eta/\partial x, \tag{2.23}$$

the latter equation reflecting that by Alfvén's theorem the sheet (or, rather, its cross-section in the x, z plane) remains a magnetic field line. Further, since all perturbation quantities in this steady-amplitude wave situation depend on z and $x-ct$ alone, (2.22) can be simplified to give

$$\int pw\,dx + U\rho_0\int u\left(\frac{\partial\eta}{\partial t}+U\frac{\partial\eta}{\partial x}\right)dx - \frac{U}{\mu}\int Bb_x\frac{\partial\eta}{\partial x}\,dx$$
$$= \int pw\,dx + U\int(\rho_0 uw - \mu^{-1}b_x b_z)\,dx, \tag{2.24}$$

whence (2.19).

Thus F represents the mean upward flux of energy at any height due to the presence of the waves only when $U=0$, or equivalently when this flux is measured by an observer moving with the local flow speed $U(z)$ at that height. The two fluxes F and $\mathscr{F}$ are, however, simply related, for defining $\theta \equiv kx-\omega t$ and using $\tilde{\psi}$ to denote the complex conjugate of $\hat{\psi}$ we may show, using (2.16), that

$$F = \overline{pw} = \overline{\tfrac{1}{4}(\hat{p}\,e^{i\theta}+\tilde{p}\,e^{-i\theta})(\hat{w}\,e^{i\theta}+\tilde{w}\,e^{-i\theta})}$$
$$= \tfrac{1}{4}(\hat{p}\tilde{w}+\tilde{p}\hat{w}) = -\tfrac{1}{4}i\rho_0(c-U)Pk^{-1}(\hat{w}'\tilde{w}-\tilde{w}'\hat{w}), \tag{2.25}$$

and on using (2.13)–(2.15) we obtain

$$\overline{pw}/(c-U) = \rho_0\overline{uw} - \mu^{-1}\overline{b_x b_z}. \tag{2.26}$$

So from (2.19)

$$\mathscr{F} = cF/(c-U). \tag{2.27}$$

We note, however, that (2.27) can alternatively be derived almost immediately from the first term of (2.22) without reference to Reynolds or Maxwell stresses. Since η is a function only of $x-ct$ we find, using (2.23), that

$$w = \frac{\partial\eta}{\partial t}+U\frac{\partial\eta}{\partial x} = -(c-U)\frac{\partial\eta}{\partial x}$$

so

$$\mathscr{F} = \int p\left(w-U\frac{\partial\eta}{\partial x}\right)dx = \int pw\left(1+\frac{U}{c-U}\right)dx = \frac{cF}{c-U}.$$

It is clear from (2.26) and (2.27) that the *net* upward energy flux $\mathscr{F}$ is simply a constant multiple of the sum of the horizontal Reynolds and Maxwell stresses, for

$$\mathscr{F} = c(\rho_0\overline{uw} - \mu^{-1}\overline{b_x b_z}).$$

Further, by multiplying (2.12) by $\tilde{w}$ and subtracting the result of multiplying its complex conjugate by $\hat{w}$ we obtain

$$d\{P(\hat{w}'\tilde{w}-\tilde{w}'\hat{w})\}/dz = 0,$$

whence, using (2.27),

$$d\mathscr{F}/dz = 0. \tag{2.28}$$

This result may be interpreted in a variety of different ways. It states that the sum of the horizontal Reynolds and Maxwell stresses (2.26) is independent of height, an extension to the hydromagnetic case of a result due to Eliassen & Palm (1961). That $\mathscr{F}$, rather than F, is constant with height in the present situation can be intimately related to the fact that when a wave packet propagates through the shear flow it is wave action [see (1.12) and cf. (2.27)] rather than wave energy that is conserved as the packet moves (see §4 and Bretherton 1966; Bretherton & Garrett 1968; Hayes 1970). Again, (2.28) simply states that the mean energy flux past a given height z_1 is precisely that past any other height z_2, since the assumptions of time-independent amplitude and no diffusive processes have ruled out (to leading order, at least) any change with time of the mean flow or magnetic field by the waves (see §4 and appendix A), so that the total mean energy between z_1 and z_2 must remain constant. These considerations, especially the last, will be helpful in the interpretation of the over-reflexion problem in §3.

Plane waves of constant amplitude

When the undisturbed flow speed U and magnetic field B (and hence Alfvén speed V) are constant with height (2.12) has solutions $\hat{w} \propto \exp imz$, where

$$m^2 = k^2(N^2/Sk^2 - 1) \tag{2.29}$$

and we have defined

$$S \equiv (c-U)^2 - V^2. \tag{2.30}$$

Clearly the vertical wavenumber m is real if and only if $0 < Sk^2 < N^2$, i.e.

$$V^2 < (c-U)^2 < V^2 + N^2/k^2. \tag{2.31}$$

Rewritten as a dispersion relation for ω, (2.29) becomes

$$(\omega - Uk)^2 = V^2k^2 + N^2k^2/(k^2+m^2), \tag{2.32}$$

and reduces to Doppler-shifted versions of the familiar dispersion relations for Alfvén and internal gravity waves when $N = 0$ and $V = 0$ respectively (e.g. Lighthill 1967). Provided m is real we can calculate the vertical group velocity

$$w_g \equiv \frac{\partial \omega}{\partial m} = \frac{-N^2km}{(c-U)(k^2+m^2)^2}, \tag{2.33}$$

and notice that when $U = 0$ these waves share with pure internal gravity waves the property of a vertical group velocity *of opposite sign* to the vertical phase velocity.

The mean wave energy E per unit volume is defined to be

$$E \equiv \tfrac{1}{2}\rho_0\overline{(\mathbf{u}^2 + \mu^{-1}\rho_0^{-1}\mathbf{b}^2 + N^2\eta^2)}, \tag{2.34}$$

where $\mathbf{u}$ and $\mathbf{b}$ denote the perturbation velocity and magnetic fields and η denotes the vertical displacement of a particle from its equilibrium position (see appendix A). It represents the energy due to the presence of 'net' waves only for an observer moving with velocity U in the x direction, and does not, when added to the energy density of the undisturbed state, give the total energy density of the disturbed system, because in setting up the steady wave train second-order mean motions can be induced which change the energy of the background state (in a quite crucial way in the over-reflexion problem of §3). By using (2.13)–(2.15) and (2.32) we find

$$E = \tfrac{1}{2}\rho_0(1+m^2/k^2)\,|\hat{w}|^2, \tag{2.35}$$

and it is also a simple matter using (2.25) to show that provided m is real

$$\overline{pw} = Ew_g = -\rho_0 N^2 m\,|\hat{w}|^2/2k(c-U)\,(k^2+m^2). \tag{2.36}$$

If m is imaginary, on the other hand, (2.25) gives $\overline{pw} = 0$ as expected for an evanescent wave.

3. Over-reflexion at a vortex–current sheet

We now consider hydromagnetic–gravity waves propagating in a fluid which is both stably and continuously stratified, while the basic flow, magnetic field and Alfvén speed take constant values U_1, B_1 and V_1 in the region $z < 0$ and constant values U_2, B_2 and V_2 in the region $z > 0$. Equation (2.12) evidently admits exponential solutions for $\hat{w}(z)$ in both regions, the vertical wavenumber in each being given by (2.29) with appropriate subscripts, and we now consider the consequences of a wave incident on the vortex–current sheet from $z = -\infty$. We write the vertical velocities associated with the incident (i), reflected (r) and transmitted (t) waves respectively as

$$\hat{w}_i = A_i e^{im_1 z}, \quad \hat{w}_r = A_r e^{-im_1 z}, \quad \hat{w}_t = A_t e^{im_2 z}, \tag{3.1}$$

where

$$m_1^2 = N^2/S_1 - k^2, \quad m_2^2 = N^2/S_2 - k^2. \tag{3.2}$$

We shall be primarily interested in the case when both m_1 and m_2 are real, and the ambiguity in their signs left by (3.2) is then removed by application of the radiation condition. Since this effectively requires consideration of how the assumed steady-state wave system would be set up a proper discussion of this, including its energetic aspects, is deferred to §4. The result is simply that we must insist that the vertical group velocity of the incident and transmitted waves be positive and that the vertical group velocity of the reflected wave be negative. Adopting (without loss of generality) the convention that k is positive, we see from (2.33) and (3.1) that this implies

$$(c-U_1)\,m_1 < 0, \quad (c-U_2)\,m_2 < 0. \tag{3.3}$$

In the case when m_2 is imaginary its sign must be such that $\hat{w}_t$ tends to zero, rather than increases without bound, as $z \to \infty$.

The interface, which is of course distorted by the wave with a displacement proportional to $\exp i\,(kx-\omega t)$, must remain a material surface, and this leads to the linearized kinematic condition

$$\frac{\hat{w}_i + \hat{w}_r}{c-U_1} = \frac{\hat{w}_t}{c-U_2} \quad \text{at} \quad z = 0. \tag{3.4}$$

Equation (2.2) then implies, as expected by Alfvén's theorem in this situation of a perfectly conducting 'frozen-in' magnetic field, that the cross-section of the interface in the x, z plane remains a magnetic field line. To avoid infinite accelerations the appropriate dynamic condition at the interface (see, for example, Shercliff 1965, pp. 64–66) is then simply continuity of total pressure p, which implies on using (2.16) that

$$\frac{S_1(\hat{w}_i' + \hat{w}_r')}{c - U_1} = \frac{S_2 \hat{w}_t'}{c - U_2} \quad \text{at} \quad z = 0. \tag{3.5}$$

From (3.1), (3.4) and (3.5) we thus find

$$\frac{A_r}{A_i} = \frac{1-Q}{1+Q}, \quad \frac{A_t}{A_i} = \frac{2(c-U_2)}{(c-U_1)(1+Q)}, \tag{3.6}$$

where

$$Q \equiv m_2 S_2 / m_1 S_1, \tag{3.7}$$

and we define the reflexion coefficient $R \equiv |A_r/A_i|$.

In discussing the reflexion of the wave at the interface we first note that we must choose c such that $V_1^2 < (c-U_1)^2 < V_1^2 + N^2/k^2$ in order that m_1 is real.

If c does *not* satisfy the inequality $V_2^2 < (c-U_2)^2 < V_2^2 + N^2/k^2$, m_2 is imaginary, the transmitted wave is evanescent, Q is imaginary, $R = 1$, and we have perfect reflexion.

If c *does* satisfy the inequality $V_2^2 < (c-U_2)^2 < V_2^2 + N^2/k^2$ the reflexion coefficient can take one of two forms, for (*a*) if $(c-U_1)(c-U_2) > 0$ then $m_1 m_2 > 0$, $Q > 0$, $R < 1$ and we have partial reflexion, while (*b*) if $(c-U_1)(c-U_2) < 0$ then $m_1 m_2 < 0$, $Q < 0$, $R > 1$ and we have over-reflexion.

We note at once that over-reflexion can take place only by virtue of the shear flow, since a necessary (but not sufficient) condition for its occurrence is

$$(c-U_1)(c-U_2) < 0, \tag{3.8}$$

and if $U_1 = U_2$ this is impossible, no matter what magnetic fields B_1 and B_2 are chosen. Equation (3.8) states simply that the horizontal phase speed must take a value between the two fluid speeds if over-reflexion is to occur.

Indeed, it turns out that given any shear flow, which in the absence of a magnetic field will over-reflect *some* waves of appropriate phase speeds c, application of a suitably large magnetic field can make over-reflexion impossible. To see this we note that in addition to (3.8) we need (2.31) to be satisfied on both sides of the interface, whence no over-reflexion can occur, *for any c*, unless

$$|V_1| + |V_2| < |U_2 - U_1| < (V_1^2 + N^2/k^2)^{\frac{1}{2}} + (V_2^2 + N^2/k^2)^{\frac{1}{2}}. \tag{3.9a, b}$$

This in turn implies that, for a given system, over-reflexion will not occur if the horizontal wavelength is too short.

The normal modes of the system

It is necessary here to consider *three*-dimensional disturbances, i.e.

$$w = \mathscr{R}[\hat{w}(z) \exp i(kx + ly - \omega t)],$$

etc., and the appropriate counterpart of (2.16) for the case when U and V are constant is

$$\hat{p} = -i\rho_0 k S \hat{w}' / (k^2 + l^2)(c - U). \tag{3.10}$$

The normal modes have structure $\hat{w}_1 = A_1 e^{im_1 z}$ and $\hat{w}_2 = A_2 e^{im_2 z}$ in regions 1 and 2 respectively, where m_1 and m_2 satisfy the dispersion relation

$$m^2 = (k^2 + l^2)(N^2/Sk^2 - 1) \tag{3.11}$$

in the respective regions, which of course reduces to (2.29) when $l = 0$.

When $c = c_R + ic_I$ both m_1 and m_2 will in general be complex, and the appropriate root to be taken in each case must be such as to ensure that the disturbances vanish as $|z| \to \infty$. Because the basic flow and magnetic fields are in the x direction only, the kinematic condition at the interface is simply

$$\hat{w}_1/(c - U_1) = \hat{w}_2/(c - U_2) \quad \text{at} \quad z = 0 \tag{3.12}$$

[cf. (3.4)], and does not involve l. Further, because l occurs only as a multiplicative factor common to both sides of the interface in the new expression (3.10) for $\hat{p}$, the continuity of total pressure across the interface simply means that

$$S_1 \hat{w}_1'/(c - U_1) = S_2 \hat{w}_2'/(c - U_2) \quad \text{at} \quad z = 0 \tag{3.13}$$

[cf. (3.5)], so that we obtain as the equation for the normal modes

$$m_1 S_1 = m_2 S_2. \tag{3.14}$$

There are three possible roots of this, namely

$$c = \frac{U_2^2 - U_1^2 - (V_2^2 - V_1^2)}{2(U_2 - U_1)} \tag{3.15}$$

(which is such that $S_1 = S_2$, and hence $m_1 = m_2$) and

$$c = \tfrac{1}{2}(U_1 + U_2) \pm 2^{-\frac{1}{2}}[N^2/k^2 + V_1^2 + V_2^2 - \tfrac{1}{2}(U_2 - U_1)^2]^{\frac{1}{2}}, \tag{3.16}$$

but the admissibility of each, inasmuch as all the conditions of the problem must be satisfied, depends in a complicated way on the parameters involved, as does the physical nature of the associated perturbation fields. We shall not discuss this at length here, but simply note that the root (3.15) is always real, while (3.16) yields exponentially growing normal modes, if k is sufficiently large, unless the magnetic fields are strong enough, i.e. unless

$$V_1^2 + V_2^2 > \tfrac{1}{2}(U_2 - U_1)^2, \tag{3.17}$$

which is precisely the condition for stability obtained by Michael (1955, 1961; see also Axford 1960) in the case when stratification is absent.

In the circumstances (3.9) for which over-reflexion may occur, the root (3.15) corresponds to two waves of equal wavelength and constant amplitude propagating away from the vortex–current sheet, one towards $z = +\infty$ and the other towards $z = -\infty$. Such a solution would ultimately be realized if one could give the vortex–current sheet an x-periodic initial normal velocity at $t = 0$, say, without disturbing the fluid elsewhere (see §5). In less simply defined regions of parameter space the roots (3.16) have similar interpretations.

Some examples of over-reflexion

Inequality (3.9 a) is easily seen to be compatible with (3.17), but together they evidently leave only a comparatively narrow band of parameter space in which over-reflexion can occur with the vortex–current sheet being at the same time stable (see figure 1). Nevertheless by suitably choosing c over-reflexion can always be found anywhere in that band.

We have plotted in figure 4 (a)–(d) the reflexion coefficient R against the horizontal phase speed c for four different values of V. Speeds have been non-dimensionalized with respect to that of the lower fluid and both the velocity and the magnetic field have been given jumps by a factor of five across the interface, thus $U_1 = 1$, $U_2 = 5$, $V_1 = V$, $V_2 = 5V$, and N/k has been chosen as 4·5 (so the horizontal wavelength has been fixed and variations in c reflect those in the frequency ω). Gaps in the graphs where no entry for R occurs correspond to values of c for which m_1 is imaginary, in which case the 'incident' wave is evanescent and has no component propagating towards the vortex–current sheet. In the absence of a magnetic field all waves with values of c between 1 and 5 are over-reflected, but as V increases from zero the band of over-reflected waves is reduced, since the magnetic field starts causing perfect reflexion at the top end and evanescence of the incident wave at the lower end. The system becomes stable once V reaches $(\frac{4}{13})^{\frac{1}{2}} = 0{\cdot}5547$, and figure 4 ($c$) shows over-reflexion still taking place for c between about 1·5 and 2·2. As V increases still further the over-reflexion band becomes even narrower, and finally disappears altogether when V exceeds $(\frac{4}{9})^{\frac{1}{2}} = 0{\cdot}6667$, as shown in figure 4 ($d$).

The reflexion coefficient is formally (since the linear analysis then breaks down) infinite in each case for a certain value of c, namely that corresponding to the natural mode of the system given by (3.15), and the amplitude of the transmitted wave is then infinite also. If a source of waves were to be switched on somewhere below the vortex–current sheet, this resonance effect would lead to a growth with time of the reflected and transmitted waves of this particular wave speed c until such time as it was terminated by nonlinear effects or the source was switched off again.† This resonance also occurs if the horizontal phase speed matches that of one of the other two natural modes of the system given by (3.16), but owing to (3.8) this can occur only if

$$(U_2 - U_1)^2 > V_1^2 + V_2^2 + N^2/k^2 \tag{3.18}$$

and this is impossible with the choices of U_1, U_2 and N/k in figures 4 (a)–(d), so that these other resonances do not appear. In figures 4 (e)–(j), however, N/k has been reduced from 4·5 to 3·0, so that (3.18) *is* satisfied for appropriately low values of V. When $V = 0$ all three resonances are observed (figure 4 e), at $c = 2{\cdot}293$, 3·000 and 3·707 according to (3.15) and (3.16), but when $V = 0{\cdot}25$ the critical values of c are 1·854, 2·812 and 4·146, the last of which falls outside the range in which (2.31) is valid, so that only a double-peaked structure is found (figure 4 g). For larger values of V, (3.18) can be violated and only a single

† A special case of this has been analysed in detail by McIntyre & Weissman (1976), who find growth as the first power of t for a constant incident wave.

resonance is observed, at the phase speed given by (3.15). Just how prominent a part is played in the reflected wave field by these resonant modes, when the incident wave takes the form of a transient disturbance containing a whole spectrum of frequencies and wavenumbers, cannot be assessed until a full initial-value analysis along the lines followed by Jones & Morgan (1972) has been carried out.

Effects of a jump in mean density across the sheet

In the analysis so far we have taken the mean density $\rho_0(z)$ to decrease slowly *and continuously* with height throughout the system. Some new features emerge if we now suppose that like U and V the density changes discontinuously across the sheet from ρ_- to ρ_+ (with $\rho_- > \rho_+$). In the Boussinesq approximation this leaves practically the whole development unchanged to leading order, provided that the fractional change in density $\Delta \equiv (\rho_- - \rho_+)/\rho_0$ is small and the density *gradients* are kept the same as before, except that the dynamic interface condition (3.4) now requires modification.

If $z = \eta(x-ct)$ is the equation of the interface, the expression for total pressure balance across it reduces, when linearized, to

$$p_i + p_r + \eta\left(\frac{dp_0}{dz}\right)_- = p_t + \eta\left(\frac{dp_0}{dz}\right)_+ \quad \text{at} \quad z = 0. \tag{3.19}$$

By virtue of the initial magnetohydrostatic balance (3.19) reduces to

$$p_i + p_r = p_t + \rho_0 g \eta \Delta \quad \text{at} \quad z = 0. \tag{3.20}$$

We have, in addition, the same kinematic conditions as before,

$$\left.\begin{aligned} w_i + w_r &= \partial\eta/\partial t + U_1\, \partial\eta/\partial x \\ w_t &= \partial\eta/\partial t + U_2\, \partial\eta/\partial x \end{aligned}\right\} \quad \text{at} \quad z = 0 \tag{3.21}$$

[whence (3.4)], and by combining (3.20) and (3.21) we have

$$\frac{S_1(\hat{w}_i' + \hat{w}_r')}{c - U_1} = \frac{S_2 \hat{w}_t' + g\hat{w}_t \Delta}{c - U_2} \quad \text{at} \quad z = 0 \tag{3.22}$$

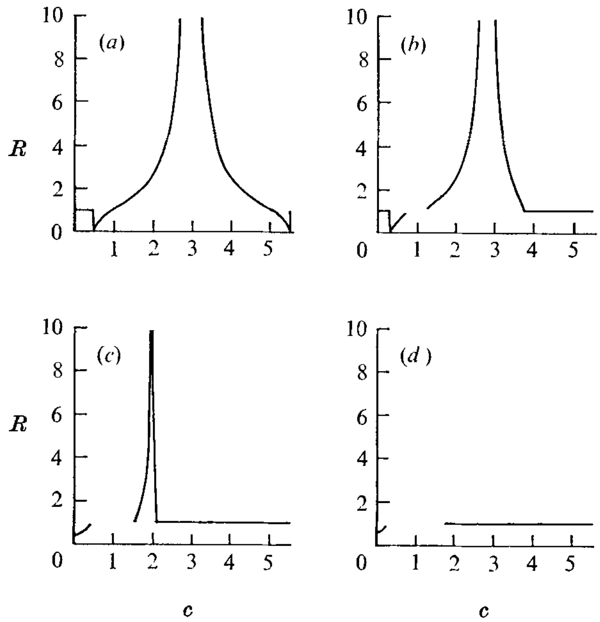
in place of (3.5). Proceeding as before we obtain

$$\frac{A_r}{A_i} = \frac{m_1 S_1 - m_2 S_2 + ig\Delta}{m_1 S_1 + m_2 S_2 - ig\Delta}. \tag{3.23}$$

For real values of the frequency ω the quantities S_1 and S_2 are, of course, real [see (2.30)]. The wavenumbers m_1 and m_2 are given as before by (3.11), and while that (m_1) corresponding to the incident wave must be real, m_2 may be imaginary, in which case there is no upward flux of energy above the interface and the reflexion coefficient R is unity, as evinced by (3.23). If m_2 is also real, on the other hand,

$$R^2 = \frac{(m_1 S_1 - m_2 S_2)^2 + g^2\Delta^2}{(m_1 S_1 + m_2 S_2)^2 + g^2\Delta^2}, \tag{3.24}$$

and the circumstances in which under- or over-reflexion will take place are then evidently precisely the same as in the absence of a density jump. We note,



FIGURES 4(*a*–*d*). For legend see facing page.

however, that in the presence of a density jump the amplitude of the reflected wave is finite for all values of the horizontal phase speed c.

For the purposes of illustration by specific examples it is convenient to measure the density jump, like everything else, by an associated *speed*, and we choose the speed $\mathscr{C}$ of the waves that would propagate (solely) along the interface if the fluids on either side were of *constant*, but slightly differing, density (ρ_- and ρ_+) and if both the basic flow and the magnetic field were absent, i.e.

$$\mathscr{C}^2 = g\Delta/2k. \tag{3.25}$$

A rough order-of-magnitude estimate of the various terms in (3.24) indicates that in either the numerator or the denominator the ratio of the first and second term will typically be $\sim c^2/\mathscr{C}^2$. Thus if $c^2 \gg \mathscr{C}^2$ the reflexion coefficient for waves with phase speed given by (3.15), though finite, will be large, $\sim |c/\mathscr{C}|$. In figures 4 (*f*) and (*h*) the reflexion coefficients are plotted, for two particular examples with $\mathscr{C} = 0{\cdot}4$, alongside their counterparts when $\mathscr{C} = 0$ (figures 4 *e*, *g*), and the limitation of the resonant peaks is evident. As the density jump across the interface, and hence $\mathscr{C}$, is increased the maximum reflexion coefficient drops quite rapidly. This is illustrated for the case $V = 0{\cdot}58$ (cf. figures 4 *a*–*d*) by figures 4 (*i*) and (*j*) and figure 5. The latter shows that if $\mathscr{C}$ is greater than about 2 the maximum reflexion coefficient is virtually indistinguishable from, but still always greater than, unity.

The interface amplitude η can be readily calculated from (3.21), and when normalized by the displacement amplitude of the incident wave displays a similarly sharp decrease as $c^2/\mathscr{C}^2$ decreases. Delisi & Orlanski (1975) have recently studied a simpler system with no basic flow or magnetic field and a *homogeneous* fluid on the 'far' side of the density-jump interface, so that $R = 1$. By an analysis

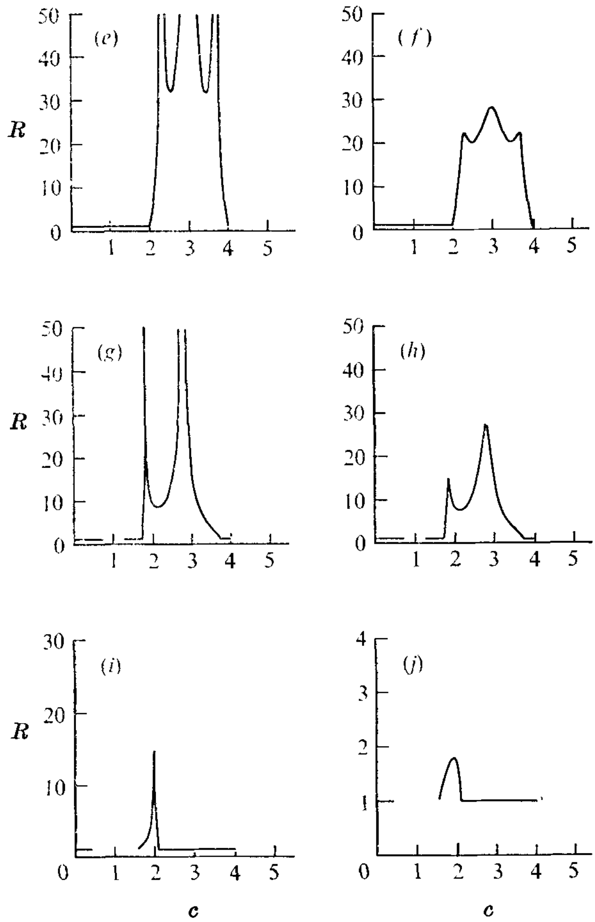


FIGURE 4. Plots of reflexion coefficient R against horizontal phase speed c in various specific cases illustrating over-reflexion (see text for full description). (*a*)–(*d*) are for $N/k = 4\cdot5$ and no density jump across the interface ($\mathscr{C} = 0$). In them V takes the values 0·00, 0·25, 0·58 and 0·75 respectively. (*e*)–(*j*) are for $N/k = 3\cdot0$, the values of V and $\mathscr{C}$ being as follows:

	(*e*)	(*f*)	(*g*)	(*h*)	(*i*)	(*j*)
V	0·00	0·00	0·25	0·25	0·58	0·58
$\mathscr{C}$	0·00	0·40	0·00	0·40	0·40	1·26

similar to that above they found that the interface amplitude is a maximum when the horizontal phase speed c matches the interfacial wave speed $\mathscr{C}$ and decreases sharply as $c^2/\mathscr{C}^2$ decreases. They conducted experiments which verified these predictions and showed, more significantly, that incident and reflected wave trains of sufficient amplitude can lead to overturning in a limited region near the interface (see their figure 8), apparently due to local gravitational instability caused by the horizontal advection of density. It will evidently be important to bear in mind this possibility of wave 'breaking' in future studies of internal gravity wave *over*-reflexion, when the amplitude of the reflected wave may be quite large even if that of the incident wave is not.

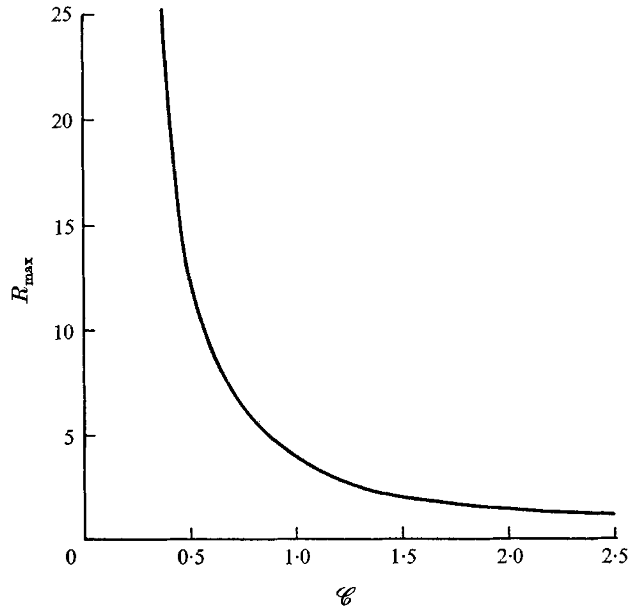


FIGURE 5. Illustrating, for the case $V = 0{\cdot}58$ and $N/k = 4{\cdot}5$, the rapid drop in the maximum reflexion coefficient as the jump in mean density across the interface, measured by $\mathscr{C}$, increases.

Energy balance across the interface

From (3.21), w_t and η are precisely out of phase, so that

$$\overline{p_t w_t}/(c-U_2) = \overline{(p_t+\rho_0 g\eta\Delta)w_t}/(c-U_2) \quad \text{at} \quad z=0,$$

and from (3.4) and (3.20) we thus find

$$\overline{p_t w_t}/(c-U_2) = \overline{(p_i+p_r)(w_i+w_r)}/(c-U_1) \quad \text{at} \quad z=0 \tag{3.26}$$

$$= (\overline{p_i w_i}+\overline{p_r w_r})/(c-U_1) \quad \text{at} \quad z=0, \tag{3.27}$$

owing to the phase relationships between the various quantities evident from (2.16) and (3.1). Equations (3.26) and (3.27) show that the net upward energy flux $\mathscr{F}$ is continuous at the interface, and thus the same everywhere. As pointed out in § 1, if we consider the lower fluid to be at rest ($U_1 = 0$) and take $U_2 = U > 0$, it is then inevitable in over-reflecting circumstances, since c is then less than U, that $\mathscr{F}$ is negative. In the next section we investigate in detail how the upward-propagating transmitted wave accomplishes this net *downward* transport of energy.

4. Evolution of a hydromagnetic wave train in a stratified fluid; changes in the mean flow

We now investigate by a multiple-scale procedure the propagation of a hydromagnetic–gravity wave train whose frequency ω and horizontal wavenumber k are constant but whose amplitude varies with height *and time* on scales very long compared with one wavelength and one period respectively (cf. Bretherton 1966, 1969*b*). We accordingly introduce the 'slow' variables

$$Z \equiv \alpha z, \quad T \equiv \alpha t, \tag{4.1}$$

where $\alpha \ll 1$ is a dimensionless measure of how slowly the wave train is modulated, inasmuch as at any given time/height its amplitude varies by a factor $O(1)$ over a height/time scale of $O(\alpha^{-1})$ wavelengths/periods. We also introduce the (subsidiary) dimensionless parameter $\beta \ll 1$, the ratio of a typical vertical wavelength to the density scale height. Since the Boussinesq approximation will be made, the following analysis for a slowly modulated wave train is self-consistent only when $\beta \ll \alpha$. While to supplement § 3 we need strictly speaking consider only the case when U and V are constant, it is instructive to consider the more general case in which U and V also vary on the *slow* height scale.

We thus consider perturbations $\mathbf{u} = (u, 0, w)$ to the basic flow $[U(Z), 0, 0]$, ρ to the density, p to the total pressure, and $\mathbf{f} = (f, 0, h)$ to the basic magnetic field $[V(Z), 0, 0]$. It should be noted that we are actually using here the associated Alfvén speeds, i.e. the magnetic fields divided by the constant (owing to the Boussinesq approximation) factor $(\mu\rho_0)^{\frac{1}{2}}$. Defining

$$\sigma = \rho g/\rho_0, \quad \phi = p/\rho_0 \tag{4.2}$$

and making *only* the Boussinesq approximation, the equations for these perturbations are

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)u + \alpha U'w - V\frac{\partial f}{\partial x} - \alpha V'h + \frac{\partial \phi}{\partial x} = \mathbf{f}.\nabla f - \mathbf{u}.\nabla u, \tag{4.3}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)w - V\frac{\partial h}{\partial x} + \sigma + \frac{\partial \phi}{\partial z} = \mathbf{f}.\nabla h - \mathbf{u}.\nabla w, \tag{4.4}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)f + \alpha V'w - V\frac{\partial u}{\partial x} - \alpha U'h = \mathbf{f}.\nabla u - \mathbf{u}.\nabla f, \tag{4.5}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)h - V\frac{\partial w}{\partial x} = \mathbf{f}.\nabla w - \mathbf{u}.\nabla h, \tag{4.6}$$

$$\partial u/\partial x + \partial w/\partial z = 0, \tag{4.7}$$

$$(\partial/\partial t + U\,\partial/\partial x)\,\sigma - N^2 w = -\mathbf{u}.\nabla\sigma, \tag{4.8}$$

where primes denote differentiation with respect to Z. [We may note that if $U*$ and $V*$ are characteristic values of U and V, λ is a characteristic wavelength and the variables $(\mathbf{x}, \mathbf{u}, \mathbf{f}, \sigma, \phi, t, U, V)$ are made dimensionless by the reference values $(\lambda, U*, V*, N^2\lambda, N^2\lambda, LU_*^{-1}, U*, V*)$, the dimensionless equations are precisely those above, except that N^2 is replaced by unity in (4.8), if we choose $U* = V* = N\lambda$, which is the parameter regime in which we are interested. This implies that the Richardson number $Ri = N^2/(dU/dz)^2$ is large, $O(\alpha^{-2})$.]

We now expand all perturbation variables ψ in powers of $\epsilon \ll 1$, a dimensionless measure of the amplitude, i.e.

$$\psi = \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + \dots. \tag{4.9}$$

The amplitude equation; conservation of wave action

The leading terms, $u^{(1)}$ etc., naturally satisfy the equations obtained from (4.3)–(4.8) by striking out the right-hand sides. To these linearized equations we seek solutions

$$\psi^{(1)} = \psi_1 + \alpha\psi_2 + \ldots, \tag{4.10}$$

where

$$\psi_n = \mathscr{R}[\hat{\psi}_n(Z,T)\exp i\{kx + \vartheta(z) - \omega t\}] \tag{4.11}$$

and a local vertical wavenumber m is defined in terms of the phase function $\vartheta(z)$ by

$$m(Z) \equiv d\vartheta/dz. \tag{4.12}$$

Substituting into the linearized versions of (4.3)–(4.8) we obtain, at order α^0,

$$(\hat{u}_1, \hat{f}_1, \hat{h}_1, \hat{\sigma}_1, \hat{\phi}_1) = \left[-\frac{m}{k}, \frac{mV}{\omega_D}, -\frac{Vk}{\omega_D}, \frac{iN^2}{\omega_D}, \left(V^2 - \frac{\omega_D^2}{k^2}\right)\frac{m}{\omega_D}\right]\hat{w}_1 \tag{4.13}$$

and

$$\omega_D^2 = V^2k^2 + N^2k^2/(k^2+m^2), \tag{4.14}$$

where ω_D denotes the Doppler-shifted wave frequency

$$\omega_D(Z) \equiv \omega - kU(Z). \tag{4.15}$$

Equations (4.13) and (4.14) merely state that the relations for a constant-amplitude plane wave derived in §2 hold *locally*. From (4.14) we see that the vertical wavenumber m will vary with height (on the slow scale Z) owing to variations with height of U and V.

At order α^1, however, we obtain

$$i(\omega_D\hat{u}_2 + Vk\hat{f}_2 - k\hat{\phi}_2) = \partial\hat{u}_1/\partial T + U'\hat{w}_1 - V'\hat{h}_1, \tag{4.16}$$

$$i(\omega_D\hat{w}_2 + Vk\hat{h}_2 - m\hat{\phi}_2) - \hat{\sigma}_2 = \partial\hat{w}_1/\partial T + \partial\hat{\phi}_1/\partial Z, \tag{4.17}$$

$$i(\omega_D\hat{f}_2 + Vk\hat{u}_2) = \partial\hat{f}_1/\partial T + V'\hat{w}_1 - U'\hat{h}_1, \tag{4.18}$$

$$i(\omega_D\hat{h}_2 + Vk\hat{w}_2) = \partial\hat{h}_1/\partial T, \tag{4.19}$$

$$i(k\hat{u}_2 + m\hat{w}_2) = -\partial\hat{w}_1/\partial Z, \tag{4.20}$$

$$i\omega_D\hat{\sigma}_2 + N^2\hat{w}_2 = \partial\hat{\sigma}_1/\partial T. \tag{4.21}$$

By eliminating all quantities with a subscript 2 among (4.16)–(4.21) we obtain a solvability condition for these equations, and using (4.13) and (4.14) this can be cast into an equation for $\hat{w}_1$ alone:

$$\omega_D(k^2+m^2)\frac{\partial\hat{w}_1}{\partial T} + m(V^2k^2 - \omega_D^2)\frac{\partial\hat{w}_1}{\partial Z} + \left[\tfrac{1}{2}(V^2k^2 - \omega_D^2)\frac{\partial m}{\partial Z} + mk^2VV' + \frac{mk^3U'V^2}{\omega_D}\right]\hat{w}_1 = 0. \tag{4.22}$$

Now the local mean wave energy density, which we here denote by ϵ^2E [cf. (2.34)], is evidently related to the local amplitude $a = |\hat{w}_1|$ by (2.35), i.e.

$$E = \tfrac{1}{2}\rho_0(1 + m^2/k^2)a^2, \tag{4.23}$$

and if we now define the local wave-action density

$$\mathscr{A} \equiv E/\omega_D \tag{4.24}$$

we find by making extensive use of the local dispersion relationship (4.14) that (4.22) can be written in a form expressing conservation of wave action:

$$\partial\mathscr{A}/\partial T + \partial(w_g \mathscr{A})/\partial Z = 0, \tag{4.25}$$

where $w_g(Z) = \partial\omega_D/\partial m$. This is of course as expected, from the work of Bretherton & Garrett (1968).

When the amplitude of the wave is independent of time (4.25) reduces to the statement that $\mathscr{A}w_g$ is independent of height, and using the relationship $Ew_g = \overline{p_1 w_1}$ [which is valid locally, as may be established from (4.13), (4.14) and (4.23), but see (2.36)] this means that $\overline{p_1 w_1}/(c-U)$ is independent of height, which we have already shown to be the case when the amplitude is steady [see (2.28)] even when no restriction is placed on how fast U and V vary over a distance of the order of a wavelength.

The wave-induced mean motion

We now investigate (4.3)–(4.8) at second order in wave amplitude, which amounts to appending superscripts (2) and (1) to perturbation quantities on the left- and right-hand sides respectively. The forcing terms on the right-hand side can be calculated from the linearized solutions above, and each has two components: a part proportional to $\exp[\pm 2i(kx+\theta-\omega t)]$, which simply forces the first harmonic of the fundamental oscillation (with zero mean), and a mean component that fluctuates only on the slow scales Z and T. We therefore assume that the mean second-order perturbations so forced also vary only with Z and T, and derive equations for these by taking the horizontal average of (4.3)–(4.8), denoted by an overbar. Thus, writing $\overline{u} = \overline{u^{(2)}}$, etc., we obtain

$$\alpha(\partial\overline{u}/\partial T + U'\overline{w} - V'\overline{h}) = \overline{\mathbf{f}^{(1)}.\nabla f^{(1)}} - \overline{\mathbf{u}^{(1)}.\nabla u^{(1)}}, \tag{4.26}$$

$$\alpha(\partial\overline{w}/\partial T + \partial\overline{\phi}/\partial Z) + \overline{\sigma} = \overline{\mathbf{f}^{(1)}.\nabla h^{(1)}} - \overline{\mathbf{u}^{(1)}.\nabla w^{(1)}}, \tag{4.27}$$

$$\alpha(\partial\overline{f}/\partial T + V'\overline{w} - U'\overline{h}) = \overline{\mathbf{f}^{(1)}.\nabla u^{(1)}} - \overline{\mathbf{u}^{(1)}.\nabla f^{(1)}}, \tag{4.28}$$

$$\alpha\, \partial\overline{h}/\partial T = \overline{\mathbf{f}^{(1)}.\nabla w^{(1)}} - \overline{\mathbf{u}^{(1)}.\nabla h^{(1)}}, \tag{4.29}$$

$$\partial\overline{w}/\partial Z = \partial\overline{h}/\partial Z = 0, \tag{4.30}$$

$$\alpha\, \partial\overline{\sigma}/\partial T - N^2\overline{w} = -\overline{\mathbf{u}^{(1)}.\nabla\sigma^{(1)}}. \tag{4.31}$$

We now envisage a wave train of the type (4.11) set up by the horizontal translation of a slightly wavy wall at $z = z_0$, say, whose corrugations are gently increasing in amplitude (from zero at time $t = 0$) on the slow time scale T, so that its equation is

$$z = z_0 + \epsilon A(T)\cos(kx - \omega t). \tag{4.32}$$

It is easily shown that $\overline{w} = 0$ at $z = z_0$, whence by (4.30), $\overline{w}$ is zero everywhere.

Similarly $\bar{h} = 0$ everywhere (essentially as a consequence of Alfvén's theorem), as is borne out by (4.29) and (4.30), since

$$\alpha \frac{\partial \bar{h}}{\partial T} = \overline{\left(f^{(1)} \frac{\partial w^{(1)}}{\partial x} - h^{(1)} \frac{\partial u^{(1)}}{\partial x} - u^{(1)} \frac{\partial h^{(1)}}{\partial x} + w^{(1)} \frac{\partial f^{(1)}}{\partial x}\right)}$$
$$= \overline{\partial(w^{(1)} f^{(1)} - u^{(1)} h^{(1)})/\partial x} = 0. \qquad (4.33)$$

Since $\bar{w} = \bar{h} = 0$, (4.28) may be written out in a similar way to (4.33), and by identical manipulations to those of Stern (1963, see his equations (4), (6) and (14)) it may be shown that (4.28) can be integrated with respect to T to give

$$\bar{f} = \frac{\alpha^2}{2k^2} \frac{\partial^2}{\partial Z^2} \left(\frac{\overline{h^{(1)2}}}{V(Z)} \right), \qquad (4.34)$$

so that $\bar{f}$ is $O(\alpha^2)$.

Turning finally to (4.26), the right-hand side can be evaluated by making use of (4.13) and (4.18)–(4.21). When $u^{(1)}$ etc. are expanded as in (4.10) the $O(1)$ contribution to the right-hand side of (4.26) vanishes, the leading one being of order α. This is composed partly of product terms involving, for example, u_1 and w_2, but we note as a further technicality that an $O(\alpha)$ contribution also emerges from terms like

$$\overline{w_1 \partial u_1/\partial z} = \tfrac{1}{4}\overline{(\hat{w}_1 e^{is} + \tilde{w}_1 e^{-is}) \partial(\hat{u}_1 e^{is} + \tilde{u}_1 e^{-is})/\partial z}$$
$$= \tfrac{1}{4}[im(\tilde{w}_1 \hat{u}_1 - \hat{w}_1 \tilde{u}_1) + \alpha(\hat{w}_1 \partial \tilde{u}_1/\partial Z + \tilde{w}_1 \partial \hat{u}_1/\partial Z)], \qquad (4.35)$$

where $s \equiv kx + \vartheta - \omega t$. We omit the remaining details of the calculation, which leads to the right-hand side of (4.26) being written (to leading order in α) as an expression involving a and $\partial a/\partial Z$ only, where $a^2 = |\hat{w}_1|^2$. On using the wave-action equation (4.25), equation (4.26) can be written as

$$\rho_0 \partial \bar{u}/\partial T = -k\, \partial(w_g \mathscr{A})/\partial Z \qquad (4.36)$$
$$= k\, \partial \mathscr{A}/\partial T \qquad (4.37)$$
$$= \frac{\rho_0(k^2 + m^2)}{2k\omega_D} \frac{\partial a^2}{\partial T}, \qquad (4.38)$$

so that $\bar{u}$, in contrast to $\bar{f}$ [see (4.34)], is $O(1)$.

Thus to leading order in α the only effect on the mean velocity and magnetic fields is a horizontal contribution to the basic velocity (being, in real terms, $O(\epsilon^2 U)$). We note that using the local relationships $Ew_g = \overline{p_1 w_1}$ and (2.26), (4.36) can be written, on returning to the notation of §2, as

$$\rho_0 \partial \bar{u}/\partial t = -\partial(\rho_0 \overline{uw} - \mu^{-1} \overline{b_x b_z})/\partial z, \qquad (4.39)$$

so that the mean acceleration is given by the gradient of the horizontal Reynolds and Maxwell stresses. We now turn to the application of the above results to the over-reflexion problem of §3, and hence take U and V as constant for most of what follows.

Energetics of over-reflexion, and the radiation condition

Since $\bar{u} = 0$ in the absence of waves, i.e. when $a = 0$, we can integrate (4.37) to give

$$\rho_0 \bar{u} = E/(c-U). \tag{4.40}$$

Thus as a finite slowly modulated wave train propagates up past any given level, the local wave energy density E will slowly increase to a maximum and (in the absence of dissipation) then diminish again to zero as the wave train passes, and the local modification $|\bar{u}|$ to the mean flow will do likewise (see figure 3), whether the mean flow is accelerated or decelerated being critically dependent, evidently, on whether c (> 0) is greater or less than U.

If the forcing (4.32) slowly reaches a constant amplitude on the time scale T and persists at that amplitude thereafter, however, the wave train will consist of a precursor (which contains $O(\alpha^{-1})$ wavelengths and whose amplitude increases with depth from effectively zero to that amplitude a_0 which the source ultimately attains) and a lower part of constant amplitude a_0 extending right down to the source. (The amplitude of this lower part, while being independent of time, will in fact vary with depth owing to the variation of ρ_0 with height, but only on a scale of $O(\beta^{-1})$ wavelengths, and such variation is neglected in the Boussinesq approximation.) When U and V are constant, so are m and w_g [see (2.33) and (4.14)], and (4.25) simply reduces to the statement that amplitude modulations propagate upwards at the group velocity; in particular, what we shall call for convenience the 'front' of the wave train (i.e. the tolerably well-defined highest point at which the amplitude is a_0) moves up at this speed. This confirms that the radiation condition in §3 has been correctly applied.

Consider now the over-reflexion problem in its simplest form ($U_1 = 0$ and $U_2 = U > 0$) slowly set up in the way described above. As the transmitted wave propagates through the upper region towards a level z_1, say, its precursor gradually sets up an alteration $\bar{u}$, given by (4.40), to the mean horizontal flow there, which remains at a constant value once the main steady-amplitude part of the wave train reaches z_1. To leading order in α no change is effected in the background magnetic field, but the mean kinetic energy density is increased by†

$$\tfrac{1}{2}\rho_0\overline{[(U+\epsilon u^{(1)}+\epsilon^2 u^{(2)}+\ldots)^2+(\epsilon w^{(1)}+\ldots)^2]}-\tfrac{1}{2}\rho_0 U^2$$
$$= \tfrac{1}{2}\rho_0 \epsilon^2(\overline{u^{(1)2}+w^{(1)2}}+2U\overline{u^{(2)}})+\ldots. \tag{4.41}$$

The first contribution in (4.41) is, of course, that from the kinetic part of the 'wave energy' (2.34), while the second represents the change in kinetic energy of the mean flow due to the waves. Thus, reverting to the notation of §§2 and 3 (i.e. replacing $\epsilon^2\bar{u}$ and $\epsilon^2 E$ by $\bar{u}$ and E), we see that when the main part of the wave train reaches any given level the total energy is enhanced by the presence of wave energy E, but that there has also been a change $\rho_0 U\bar{u}$ in the kinetic

† Here ρ_0 represents the mean density of the fluid, according to the Boussinesq approximation, and it may be confirmed that if the density in (4.41) were to be expanded in the same way as the other variables it would lead to additional terms only $O(\alpha\beta)$ compared with those in (4.41). It is necessary to know that $\bar{\sigma}$ is at most $O(\alpha)$ to conclude this, which follows from (4.31) since the right-hand side turns out to be $O(\alpha^2)$, as in Bretherton (1969*b*).

energy of the mean flow. By using (4.40) this latter contribution is evidently

$$UE/(c-U) \tag{4.42}$$

and is both negative and greater than E in magnitude if $c < U$ and over-reflexion is occurring. The *net* upward energy flux is obtained by adding the wave energy E to (4.42) and multiplying by w_g, whence we obtain

$$\mathscr{F} = cEw_g/(c-U), \tag{4.43}$$

which is precisely (2.27).

We may note that this finally clears away one cloud which hung over our application of the radiation condition, i.e. the *net* flux of energy being *downward* in the upper medium. It is now clear that in no sense is this energy at any stage in the setting up of the wave system in §3 coming 'from infinity': rather, it is at any time coming from the mean flow near the tip of the wave train in the upper region.

5. Discussion

It is intuitively clear that we can expect the detailed predictions of the analysis of over-reflexion at a vortex–current sheet in §3 and their explanation in terms of the specific wave/mean-flow interaction illustrated in figure 3 to apply in the case of a *finite* shear layer of depth d only if $d \ll \lambda$, a typical vertical wavelength. That they then do so gains support from an analysis of the non-hydromagnetic case by Eltayeb & McKenzie (1975), whose results tally with those of the corresponding vortex-sheet analysis (McKenzie 1972) as $d/\lambda \to 0$, despite the fact that there is inevitably a critical level where $c = U$ imbedded in the shear layer (since $U_1 < c < U_2$ for over-reflexion). Equation (2.28) does not necessarily hold at levels where the governing equation (2.12) is singular, and except in the limit $d/\lambda \to 0$ we must expect some discontinuity in $\mathscr{F}$ (which should, however, be small if $d/\lambda \ll 1$) and some revision of our interpretation of the over-reflexion mechanism. At the opposite extreme $d \gg \lambda$ a WKB analysis of the kind in Acheson (1973) certainly leads us to expect that the waves will instead be almost entirely absorbed within the shear layer at a level at which the Doppler-shifted wave speed equals the local Alfvén speed. Much remains to be done to clarify this and the intermediate case $d \sim \lambda$, however, and we cut short further speculation about this particular system to make some rather more general points.

We draw attention first to some fundamental differences between the mechanism of over-reflexion (when $d \ll \lambda$) and the process of critical-layer absorption. Considering the latter, we recall that, when the Richardson number†

$$Ri = N^2/(dU/dz)^2 \tag{5.1}$$

† We note that, while the parameters Ri and d/λ are, of course, in general independent, they are intimately related in a 'typical' situation under consideration here. If the layer is of constant shear and the mean flow is, say, zero below it and U above, then $Ri = N^2d^2/U^2$. If we are considering an incident wave with roughly comparable horizontal and vertical wavenumbers ($k \sim m$), if its critical level is somewhere reasonably central in the shear layer ($\omega \sim Uk$), and supposing any magnetic fields (if present at all) to be such that $V \sim U$ (in order that both (1.1) and (1.2) are satisfied), the dispersion relationship (2.32) then implies that typically $m^2 \sim N^2/U^2$. Thus typically $Ri \sim (d/\lambda)^2$ in the circumstances envisaged here.

exceeds $\frac{1}{4}$ in the pure internal gravity wave problem, a monochromatic (single c) upward-propagating steady wave train has its associated Reynolds stress $\rho_0\overline{uw}$, and hence $\mathscr{F}$ [$= c\rho_0\overline{uw}$, see (2.26) and (2.27)], suddenly attenuated across the critical level where $c = U$ [below and above which they are both constant, see (2.28)] by a factor

$$\exp\left[-2\pi(Ri-\tfrac{1}{4})^{\frac{1}{2}}\right] \tag{5.2}$$

(Booker & Bretherton 1967). The wave thus loses its total energy (including the $\rho_0 U\bar{u}$ contribution) to the mean flow at that level, or rather to a thin layer surrounding it and determined by one or more of (*a*) diffusive effects (Hazel 1967; Baldwin & Roberts 1970), (*b*) nonlinear effects, which can also lead to significant reflexion rather than absorption (Breeding 1971; Maslowe 1972; Kelly & Maslowe 1970), and (*c*) the fact that the wave will never be quite monochromatic, so that there will be a spread in its values of c and hence in the locations of the corresponding critical levels (Booker & Bretherton 1967; Lindzen & Holton 1968). The mean flow in that neighbourhood, but not elsewhere, will steadily increase if the wave source is maintained, and it will do so indefinitely if no account is taken of (*a*), (*b*) or (*c*) above. Eventually it may alter the mean flow such that the position of the critical level significantly changes, and subsequent developments may include the descent of the entire shear layer, as noted by Lindzen & Holton (1968) in their original theory of the quasi-biennial oscillation of the tropical stratosphere (see also Jones & Houghton 1971).

Now contrast this mechanism with that operating in the over-reflecting system analysed in this paper, which may be expected to be typical of others provided $d \ll \lambda$. The *extraction* of energy from the mean flow for the over-reflexion is maintained not by cumulatively slowing down a particular layer of fluid, but rather by slowing down by a fixed second-order amount [see (4.40)] a portion of the upper region which becomes deeper and deeper as time goes on, so long as the wave source is maintained.

Another distinction to be made between over-reflexion *when* $d \ll \lambda$ and critical-layer absorption is that in the latter case the alteration to the mean flow is *permanent*, i.e. persists after the wave source has been switched off and waves are no longer approaching the critical layer, while in the $d \ll \lambda$ over-reflecting regime the mean flow will at any time be significantly different (by a second-order amount) from its original value only wherever the waves have significant amplitude at that time (see figure 3). This last statement would not be true, however, if dissipation were present (e.g. Bretherton 1969*a*, 1971; Holton & Lindzen 1972; Lindzen 1973; Holton 1974); as a wave train of finite length passes any given level some of its energy would be left behind in the form of a modification to the mean flow and some would be degraded into heat.

By way of an immediate corollary to the above remarks, we interject at this stage a comment on an interesting recent proposal by Lindzen (1974) that internal gravity waves radiating from an unstable shear layer in the earth's atmosphere may be at least as important as the Kelvin–Helmholtz mechanism itself in explaining the observations of clear-air turbulence. The mathematical model is precisely that of § 3, with (of course) hydromagnetic effects removed, and Lindzen confines attention to the normal modes (3.15) and (3.16), which then simplify to

$$c = \tfrac{1}{2}(U_1 + U_2), \tag{5.3}$$

$$c = \tfrac{1}{2}(U_1 + U_2) \pm 2^{-\frac{1}{2}}[N^2/k^2 - \tfrac{1}{2}(U_2 - U_1)^2]^{\frac{1}{2}}. \tag{5.4}$$

From the latter we see that the system is always unstable to disturbances of suitably short wavelength, and it can be shown that such disturbances tend to smooth out the original velocity profile in roughly the way indicated (very schematically) in figures 6 (*a*)–(*c*). The normal mode (5.3) represents two waves of constant amplitude and equal vertical wavelength propagating away from the vortex sheet, one upwards and the other downwards (as discussed in § 3). Lindzen also notes that by the radiation condition $F = \overline{pw}$ must be positive in the upper region and negative in the lower region. He goes on to infer, however, that this implies a net extraction of energy from the vicinity of the sheet in such a way that the profile will be smoothed out roughly as shown in figures 6 (*d*)–(*f*), estimates the time taken for the profile to develop a gradient such that Ri is increased to $\frac{1}{4}$, and compares this with an estimate for the time taken for the Kelvin–Helmholtz mechanism to achieve the same end. From the results discussed above, however, it is a gradient not of F but of $\mathscr{F}$, or alternatively of wave-action flux, that is crucial to such *cumulative* changes with time of the mean flow at any level [see (4.36) and (4.39)]. Since, by (5.3), c lies between U_1 and U_2, although the signs of F are opposite in the two regions the signs of $\mathscr{F}$ are the same [see (1.5)], indeed $\mathscr{F}$ is continuous across the sheet and independent of height in the two regions as far as the tips of the two wave trains (see § 4). As discussed above, the waves represented by (5.3) lead to *no* cumulative smoothing, and as they propagate away from the vortex sheet and fill more and more of each region their effect on the mean flow is as shown in figures 6 (*g*)–(*i*). The cumulative smoothing envisaged by Lindzen can result in this model system only from the systematic growth of the Kelvin–Helmholtz instabilities, for then wave-action flux is no longer conserved. (Large-amplitude waves might provide another mechanism, but we are not equipped to comment on that here. The weakly nonlinear problem has, however, been investigated by Grimshaw 1972, 1976.)

These observations in no way affect some of the other proposals in Lindzen's (1974) paper concerning internal gravity waves and clear-air turbulence: in particular the possibility that the waves can reach large amplitude by successive over-reflexions at the shear layer, with (in principle) a perfect reflexion at the ground between each one. We take up the question of multiple over-reflexion at the end of this section.

Next, however, we must emphasize that our remarks above about the way in which over-reflexion works have been confined to the case $d \ll \lambda$. In other circumstances it would appear that the additional energy needed for the over-reflexion can be drawn steadily from a critical layer. We cite first the numerical study by Jones (1968) of internal gravity waves incident on a finite layer of constant shear separating two uniform streams. A necessary condition for over-reflexion to occur is that $U_1 < c < U_2$, in which case the wave will have a critical level in the shear zone. It is convenient to divide all such horizontal phase speeds c into two classes,

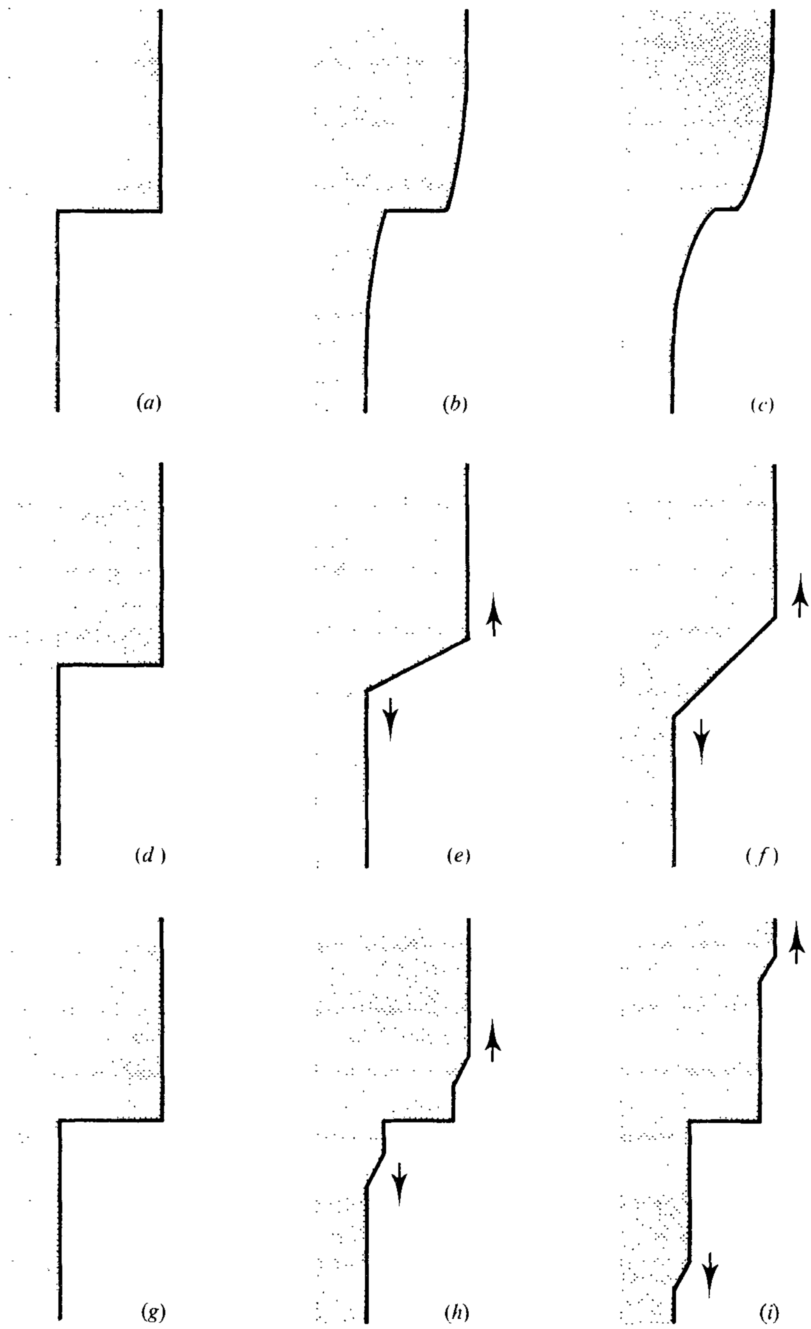


FIGURE 6. The first time sequence (*a*)–(*c*) illustrates schematically changes in the mean flow due to a Kelvin–Helmholtz instability mode on a vortex sheet in a continuously stratified fluid. The second time sequence (*d*)–(*f*) and the third time sequence (*g*)–(*i*) illustrate the changes in mean flow due to the radiation of neutral internal gravity waves from the sheet as envisaged by Lindzen and as indicated by the analysis of this paper respectively.

the first being such that the disturbance above the shear layer takes the form of an upward-propagating wave and the second being such that the disturbance above the shear layer is evanescent. Jones finds that waves with suitable speeds in the first class will be over-reflected provided $Ri < 0{\cdot}115$, a figure with which Eltayeb & McKenzie (1975) obtain close agreement (0·1129) in their analytical study of the problem. As noted above, Eltayeb & McKenzie show that as $Ri \to 0$ the reflexion coefficient tends to that obtained from a vortex-sheet analysis, in which limit we infer that the excess reflected energy is drawn from the flow above the shear layer. However, for $Ri = 0{\cdot}1$, for example, it may well be that part of the excess reflected energy is being drawn from within the shear layer, in particular from the critical level. More significant, we note that Jones also finds that waves with speeds such that the transmitted disturbance is *evanescent* can be over-reflected, provided the less stringent requirement $Ri < 0{\cdot}25$ is met. In this case $\mathscr{F}$ must be zero above the shear layer, and the over-reflexion mechanism discussed in this paper cannot be operating at all (we accordingly anticipate that over-reflexion of waves in this second class must disappear altogether in the limit $Ri \to 0$). The excess reflected energy must presumably be wholly obtained by cumulatively decelerating the flow at the critical level.

A similar situation prevails in the case of Rossby waves propagating in a zonal shear flow (Geisler & Dickinson 1974). There incident waves of *any* phase speed that are over-reflected must be evanescent on the 'far' side of the shear layer, owing to the requirement $U_1 < c < U_2$ together with the fact that Rossby waves with real north–south wavenumber are constrained, by their well-known dispersion relationship, to have westward phase propagation relative to the local flow. Again the excess reflected energy must be being cumulatively extracted from the mean flow at the critical level.

A notable feature of the numerical study by Geisler & Dickinson is the way in which periods of critical-level absorption alternate with periods of over-reflexion in an approach to a final steady state. The potential-vorticity gradient $\beta * \equiv \beta - d^2U/dy^2$ was initially positive everywhere, and Rossby waves were then switched on and suffered absorption at the critical level. The local mean flow was changed by this in such a way that $\beta*$ became negative near the critical level. This led to over-reflexion rather than absorption, and the changes so wrought in the local mean flow then led $\beta*$ to become positive again. This cycle repeated a number of times with diminishing amplitude in the approach to a steady state of perfect reflexion.

One interesting question to which the answer is not yet known is whether over-reflexion is possible (as is critical-layer absorption) in systems in which there is no mean flow. The present author believes the answer to be 'no', reflecting a further fundamental difference between the basic mechanics of the two processes. Certainly this is the case for the primary model of this paper – it was solely from the mean flow and not the magnetic field that the over-reflected wave drew its excess energy – and a current sheet alone does not suffice for over-reflexion in a number of other hydromagnetic systems which the author studied in looking for one with the desired stability properties. Here again, however, a finite-layer model with a hydromagnetic critical level at some height where the magnetic field attains a special value may conceivably lead to over-reflexion by a totally different energy transfer process (cf. the three paragraphs immediately above). Some indication that this is *not* the case arises from the fact that the very form of the attenuation factor (5.2) associated with critical-layer absorption of internal gravity waves in a shear flow reveals the need for a complete re-working and re-interpretation of the analysis when $Ri < \frac{1}{4}$, in which case over-reflexion may occur, while the corresponding attenuation factors in the two known examples of critical-layer absorption without a shear flow have no such property of becoming, formally, complex for certain parameter values (see Acheson 1972, equations (3.7) and (3.9); Acheson 1973, equation (2.12)).

We turn finally to the question of multiple over-reflexion. As Lindzen (1974) and Eltayeb & McKenzie (1975) point out, the presence of a solid boundary at which, in an idealized model, the over-reflected wave is returned to the vortex sheet by means of a perfect reflexion will inevitably lead to systematic growth of the wave with time. It is also intuitively evident that even in the absence of a solid boundary the presence of another sheet at some other level will have similar consequences, provided that the *product* of the reflexion coefficients exceeds unity for suitable phase speeds c, the growth rates being small, however, if the sheets are separated by many vertical wavelengths. This is quantitatively confirmed by the analysis in appendix B, which, following that of Berman & Ffowcs Williams (1970) for the instability of a compressible jet, shows that when the separation distance is large the (small) growth rates of normal modes with such phase speeds are just those obtained by making more precise the intuitive argument above about the systematic growth via multiple over-reflexion.

As a conceptually even simpler example we finally consider the effects on the model system studied in §3 of placing a horizontal rigid lid at some level above the vortex–current sheet, and suppose that the fluid is moving to the right with speed U above the sheet and is at rest below. There is a steady-state wave solution to the problem of perfect reflexion, because of the perfect reflexion of the transmitted wave at the boundary, but this state would never be attained because even if (1.1) is satisfied this system is unstable. After the incident wave train has penetrated the vortex–current sheet from below, thus setting up (if the phase speed c is appropriate) an over-reflected wave, its transmitted part will be perfectly reflected at the boundary, will then be over-reflected from the sheet back to the boundary and so on. At each subsequent over-reflexion the amplitude will be larger than that at the preceding one, and this will be true also of the downward-propagating transmitted wave that will accompany each such event. In this way the amplitudes get steadily larger as time goes on, and the effects of the increasing number of superposed wave trains in each region, each bigger than the last, *will* give rise to cumulative changes with time in the mean flow, that in the upper region being steadily diminished (for its kinetic energy is the source of the over-reflexion) and that in the lower region being steadily increased until, we may suppose (though this is well beyond the scope of our small-disturbance theory), the basic state is modified to a point where over-reflexion ceases.

This paper was presented at the 17th British Theoretical Mechanics Colloquium at U.M.I.S.T. in April 1975, and I am most grateful to a number of the participants for useful comments. I wish especially to thank Dr R. Grimshaw, Dr D. G. Andrews, Dr M. E. McIntyre and Dr M. A. Weissman for their subsequent help. I am also grateful to Dr R. N. Franklin for discussions on 'negative-energy' waves in plasma physics, to Mr M. Gibbons for performing the numerical computations, and to two anonymous referees for their constructive comments.

Appendix A. The energy equation for the waves and the mean flow

To establish this we start here from the full basic equations (2.1)–(2.5) and denote the basic undisturbed fields by $\mathbf{U}(z)$, $\mathbf{B}(z)$, $p_0(z)$ and $\rho_0(z)$, the total perturbations to them (i.e. including contributions from all orders in ϵ) being denoted by $\mathbf{u}$, $\mathbf{b}$, p and ρ. We shall use z^* to denote the original height of a fluid element that is currently at the height z, so that

$$w = Dz/Dt, \quad Dz^*/Dt = 0 \tag{A 1}$$

and $\eta = z - z^*$ denotes vertical displacement. In view of (2.5) elements retain their density as they move, so that the density of a displaced element is given by $\rho_0(z^*)$, and then $\rho = \rho_0(z^*) - \rho_0(z)$. By subtracting off the magnetohydrostatic balance (2.7) from (2.1), and multiplying scalarly by $\mathbf{U}+\mathbf{u}$, we obtain

$$\tfrac{1}{2}\rho_0(z^*)\frac{D}{Dt}(\mathbf{U}+\mathbf{u})^2 = -\nabla.[p(\mathbf{U}+\mathbf{u})] + \frac{1}{\mu}(\mathbf{U}+\mathbf{u}).[(\mathbf{B}+\mathbf{b}).\nabla(\mathbf{B}+\mathbf{b})] - \rho g\frac{D\eta}{Dt}. \tag{A 2}$$

In displacing a fluid element from z^* to z work is being done at each intermediate height against a net downward force ρg per unit volume, arising from the excess density of the element over that of its surroundings, so that the change H in potential energy per unit volume due to the waves is

$$\begin{aligned} H &= \int_{z^*}^{z} [\rho_0(z^*) - \rho_0(z)]\, g\, dz = \int_{z^*}^{z}\left[g\rho_0(z^*) + \frac{dp_0}{dz}\right]dz \\ &= g\eta\rho_0(z-\eta) + p_0(z) - p_0(z-\eta) \\ &= g\eta[\rho_0(z) - \eta\rho_0'(z) + \dots] - [-\eta p_0'(z) + \tfrac{1}{2}\eta^2 p_0''(z) + \dots] \\ &= -\tfrac{1}{2}g\eta^2\rho_0'(z) + \dots = \tfrac{1}{2}\rho_0 N^2\eta^2 + \dots \end{aligned} \tag{A 3}$$

[cf. (2.34)]. With some manipulation of the hydromagnetic term and the use of (2.2)–(2.5), (A 2) can then be written in the form

$$\begin{aligned} &\partial[\tfrac{1}{2}\rho_0(z^*)(\mathbf{U}+\mathbf{u})^2 + \tfrac{1}{2}\mu^{-1}(\mathbf{B}+\mathbf{b})^2 + H]/\partial t \\ &= -\nabla.[\{p + \tfrac{1}{2}\rho_0(z^*)(\mathbf{U}+\mathbf{u})^2 + \tfrac{1}{2}\mu^{-1}(\mathbf{B}+\mathbf{b})^2 + H\}(\mathbf{U}+\mathbf{u}) \\ &\qquad - \mu^{-1}\{(\mathbf{U}+\mathbf{u}).(\mathbf{B}+\mathbf{b})\}(\mathbf{B}+\mathbf{b})] \end{aligned} \tag{A 4}$$

or, equivalently,

$$\begin{aligned} &D[\tfrac{1}{2}\rho_0(z^*)(\mathbf{U}+\mathbf{u})^2 + \tfrac{1}{2}\mu^{-1}(\mathbf{B}+\mathbf{b})^2 + H]/Dt \\ &= -\nabla.[p(\mathbf{U}+\mathbf{u}) - \mu^{-1}\{(\mathbf{U}+\mathbf{u}).(\mathbf{B}+\mathbf{b})\}(\mathbf{B}+\mathbf{b})]. \end{aligned} \tag{A 5}$$

Equation (A 5) confirms that (2.20), and hence (1.5), represents the net flux of energy across a distorted (originally horizontal) material surface, since by Alfvén's theorem its cross-section remains a magnetic field line, so that $(\mathbf{B}+\mathbf{b}).\mathbf{n} = 0$.

Having dealt with the buoyancy term appropriately in (A 2) we now make the Boussinesq approximation, replacing $\rho_0(z^*)$ in (A 4) and (A 5) by some mean density which we simply denote by ρ_0, and take the horizontal average of (A 4) to obtain

$$\partial(E+\rho_0 U\bar{u}+\mu^{-1}B\bar{b}_x)/\partial t$$
$$= -\partial[\overline{pw}+U(\rho_0\overline{uw}-\mu^{-1}\overline{b_x b_z})+\mu^{-1}B(\overline{b_x w-b_z u})]/\partial z. \quad (\text{A } 6)$$

Some other terms which arise in the brackets on the right-hand side, proportional to $\bar{w}$ or $\bar{b}_z$ and apparently also of second order, vanish when $\bar{w} = \bar{b}_z = 0$, as is the case when the (steady or weakly unsteady) wave system is set up as in §4 [see (4.32)]. The relationship of (A 6) to (2.18)–(2.24) is evinced by the fact that $\overline{b_x w - b_z u}$ is precisely zero in that steady situation [from (2.13)–(2.15)], and the left-hand side of (A 6) vanishes, whence $d\mathscr{F}/dz = 0$ [see (2.19) and (2.28)].

In the weakly *unsteady* wave analysis with slowly varying U and V in §4, $\mu^{-1}B(\overline{b_x w-b_z u})$ does not vanish, but may be shown (using the linearized analysis of that section) to be $O(\alpha^2)$ compared with the dominant terms of (A 6), as is $\mu^{-1}B\bar{b}_x$ [see (4.34)]. The wave energy equation pertinent to that situation is thus

$$\partial(E+\rho_0 U\bar{u})/\partial t = -\partial[\overline{pw}+U(\rho_0\overline{uw}-\mu^{-1}\overline{b_x b_z})]/\partial z, \quad (\text{A } 7)$$

which simply states that the rate of change of *total* energy (waves + mean flow) is equal to the gradient of the *net* (rather than 'wave') upward energy flux. To complete the link of the conservation of energy here with §4 and the conservation of wave action (4.25) we see that by using (4.24) and (4.37) the left-hand side of (A 7) can be written as

$$\partial[k(c-U)\mathscr{A}+Uk\mathscr{A}]/\partial t = ck\,\partial\mathscr{A}/\partial t. \quad (\text{A } 8)$$

Using (2.26) and (2.36), which are valid locally here, the right-hand side of (A 7) can be written as

$$-\frac{\partial}{\partial z}\left[\frac{c\overline{pw}}{c-U}\right] = -\frac{\partial}{\partial z}\left[\frac{cEw_g}{c-U}\right] = -ck\frac{\partial}{\partial z}(w_g\mathscr{A}). \quad (\text{A } 9)$$

Put in this way, the two sides of (A 7) being equal is an expression of the conservation of wave action (4.25).

Appendix B. Two vortex–current sheets in a stratified fluid; instability as a result of multiple over-reflexion

In support of some remarks made in §5 we briefly consider here the stability of *two* vortex–current sheets in a stratified fluid, one at $z = -H$ and the other at $z = H$. As in the primary model considered in this paper the density is taken as a function gently and continuously decreasing with height throughout the system in such a way that the buoyancy frequency N is constant. To investigate the stability of the system we obtain an equation for the normal modes by writing

$$\hat{w} = \begin{cases} A_1\, e^{im_1 z} & \text{in region 1} \quad (z < -H), \\ A_2^+\, e^{im_2 z} + A_2^-\, e^{-im_2 z} & \text{in region 2} \quad (-H < z < H), \\ A_3\, e^{im_3 z} & \text{in region 3} \quad (z > H), \end{cases} \tag{B 1}$$

and denote the values of the basic flow speed U and Alfvén speed V in each region by suitable subscripts in the obvious way. By ensuring the continuity of

$$\hat{w}/(c-U), \quad S\hat{w}'/(c-U) \tag{B 2}$$

[see (3.4) and (3.5)] at both $z = -H$ and $z = H$ we obtain four homogeneous equations for the four amplitude constants appearing in (B 1), and these have a non-trivial solution only if

$$e^{-4im_2 H} = \left(\frac{m_1 S_1 + m_2 S_2}{m_1 S_1 - m_2 S_2}\right)\left(\frac{m_3 S_3 - m_2 S_2}{m_3 S_3 + m_2 S_2}\right). \tag{B 3}$$

Recalling from (2.29) and (2.30) that

$$m^2 = N^2/S - k^2 \tag{B 4}$$

and

$$S \equiv (c-U)^2 - V^2, \tag{B 5}$$

(B 3) yields, in principle, the eigenvalues c (or equivalently ω), which we anticipate may be complex. As in the primary model of § 3, there is an ambiguity left by (B 4) in the signs of m_1, m_2 and m_3 which must be resolved in the course of solving (B 3) for c.

In practice, however, the complexity of (B 3) is such that this task would appear extremely difficult, and we confine attention in what follows to an exploration of those modes which have very small growth rates, so that the imaginary parts of c and S are very much smaller than the corresponding real parts. This will then also be true of m if, as we suppose, the basic parameters of the system are such that modes of slow growth rate with the real part of c satisfying (2.31) in all three regions can be found. Applying the radiation condition as in § 3 to regions 1 and 3, and retaining the convention $k > 0$, we have

$$m_1(c-U_1) > 0, \quad m_3(c-U_3) < 0, \tag{B 6}$$

where it is understood that the real parts of m and c are being taken. In region 2 the way in which the ambiguity in the sign of m_2 is resolved is purely a matter of personal preference. We choose to view the contribution $A_2^+\, e^{im_2 z}$ as an upward-travelling wave and the other contribution as a downward-travelling wave, in keeping with the notation of § 3, whence

$$m_2(c-U_2) < 0. \tag{B 7}$$

We now note, by comparison with (3.6) and (3.7) and with due regard to the signs involved, that the modulus of the first factor on the right-hand side of (B 3) is the reflexion coefficient (R_-) for the downward-travelling wave $A_2^-\, e^{-im_2 z}$ incident from region 2 on the lower vortex–current sheet. Similarly the modulus of the second factor on the right-hand side of (B 3) is the reflexion coefficient (R_+) for the upward-travelling wave $A_2^+ e^{im_2 z}$ incident from region 2 on the upper sheet. It is then instructive to take the logarithm of both sides of (B 3), and then to take the real parts of both sides to obtain an expression for the imaginary part of m_2:

$$m_{2I} = (4H)^{-1} \log (R_- R_+). \tag{B 8}$$

The imaginary part of m can be related to the imaginary part of c by expanding the right-hand side of (B 4) for small c_I and taking the imaginary parts of both sides, whence

$$m_I \doteqdot -c_I N^2(c-U)/mS^2, \tag{B 9}$$

real parts of c, m and S being understood unless a subscript I is present. We note as a check on our application of the radiation condition (B 6) that it implies, via (B 9), that at any given time growing modes decay with distance upwards from the sheet at $z = H$ and decay with distance downwards from the sheet at $z = -H$, as required.

Combining (B 9), as applied to region 2, and (B 8), we have

$$\omega_I = \frac{-kS_2^2 m_2}{4HN^2(c-U_2)} \log (R_- R_+). \tag{B 10}$$

In view of (B 7) and the convention $k > 0$, ω_I is clearly positive or negative, corresponding to growth or decay of the mode with time, according as the product $R_- R_+$ is greater or less than unity. This is most easily understood by considering a wave train with prescribed (real) ω and k generated at $z = 0$ so as to propagate upwards towards $z = H$. It will be reflected there, will propagate downwards to $z = -H$ to be reflected again and so on, passing $z = 0$ on the upward journey after n such 'cycles' with its amplitude changed by a factor $(R_- R_+)^n$. If the horizontal phase speed of the wave is such that $R_- R_+ > 1$ (which means that *at least* one of the vortex–current sheets is over-reflecting it) then it will grow in amplitude with time. This argument may be related more quantitatively to the result (B 10) of the normal-mode stability analysis by noting that by combining (2.33), (B 4) and (B 10) we have

$$\omega_I = (|w_g|/4H) \log (R_- R_+), \tag{B 11}$$

where $|w_g|$ is the magnitude of the vertical component of the group velocity of the wave train in region 2. This means that according to the normal-mode theory a mode will have amplified after a time t by a factor $e^{\omega_I t}$, i.e.

$$(R_- R_+)^{|w_g| t/4H}, \tag{B 12}$$

and this is precisely what our argument above concerning multiple over-reflexion predicts, bearing in mind that after n such 'cycles' a time $4Hn/|w_g|$ will have elapsed.

It must be borne in mind that the analysis following (B 3) is approximately valid only for modes whose growth rates are small, and this precludes those with phase speeds c near to the resonance points of either sheet [see (3.15) and (3.16)]. For modes other than these we see from (B 11) that since, roughly speaking, $w_g \sim \omega/m$, the ratio of growth rate to frequency will be comparable with that of vertical wavelength to sheet separation distance, and the former will thus be small (as supposed) if the vortex–current sheets are separated by many vertical wavelengths, as is intuitively clear if the instability is viewed as being the result of multiple over-reflexion.

Appendix C. Magneto-acoustic over-reflexion at a vortex sheet

All the results needed to establish the existence of stable over-reflecting regimes (see figure 2) in this case can be drawn from the studies of Fejer (1963, 1964), McKenzie (1970) and Duhau & Gratton (1973), but we summarize the pertinent ones here to show what the various curves making up figure 2 mean. We confine attention to the case of constant density, sound speed and Alfvén speed.† In the case when U varies continuously with height we have carried out an analysis similar to that in §2: it goes through in much the same way with, in particular, (2.18), (2.19) and (2.26)–(2.28) retaining their validity and significance. When U is constant the dispersion relationship is, in the two-dimensional case,

$$(\omega - Uk)^4 - (a^2 + V^2)(k^2 + m^2)(\omega - Uk)^2 + a^2V^2k^2(k^2 + m^2) = 0 \tag{C 1}$$

(see, for example, Shercliff 1965, p. 231), i.e.

$$m^2 = \frac{k^2[(c-U)^2 - a^2][(c-U)^2 - V^2]}{(a^2+V^2)(c-U)^2 - a^2V^2}. \tag{C 2}$$

When m is real the vertical group velocity is given by

$$w_g = \frac{m[(a^2+V^2)(c-U)^2 - a^2V^2]^2}{k(c-U)^3[a^2\{(c-U)^2 - V^2\} + V^2\{(c-U)^2 - a^2\}]}. \tag{C 3}$$

For over-reflexion we need m real on both sides of the vortex sheet, and from (C 2) and (C 3) we thus obtain the following alternatives: *either*

$$(c-U)^2 > \max(a^2, V^2), \quad w_g mk(c-U) > 0 \tag{C 4}$$

or

$$\min(a^2, V^2) > (c-U)^2 > a^2V^2/(a^2+V^2), \quad w_g mk(c-U) < 0. \tag{C 5}$$

In either case we see that

$$w_g mk(c-U)S > 0. \tag{C 6}$$

In the reflexion problem the interface conditions are again continuity of p and vertical displacement, and the reflexion coefficient is

$$R = \left|\frac{m_2 S_1 - m_1 S_2}{m_2 S_1 + m_1 S_2}\right| \tag{C 7}$$

† It must be said that this special case is of little interest, unfortunately, in connexion with one of the main applications of the magneto-acoustic problem, namely to the earth's magnetopause. McKenzie (1970), for example, uses in his stability analysis the simplifying approximations $a \gg V$ on one side of the sheet (the shocked solar wind) and $a \ll V$ on the other (the magnetosphere) as being representative of local conditions. We note also here that there is a good case for blending the problems of hydromagnetic internal gravity waves and magneto-acoustic waves in connexion with the possible action of such waves as agents for the upward transfer of energy from the sun's convection zone to provide coronal heating (e.g. Lighthill 1967), although it must then be anticipated that the additional complexities involved will render fusion of over-reflexion and stability results like that achieved in figures 1 and 2 extremely difficult.

[cf. (3.6) and (3.7)]. Provided m_1 and m_2 are real, over-reflexion occurs if $m_1 m_2 S_1 S_2 < 0$, and since incident and transmitted waves must have w_g of the same sign it follows from (C 6) that we need

$$(c-U_1)(c-U_2) < 0. \tag{C 8}$$

After defining the parameters A and W as in (1.3) and some algebra involving (C 4), (C 5) and (C 8) it is possible to delineate three parameter regimes in which waves of suitable phase speed c will be over-reflected. The first, for which (C 4) is satisfied on both sides of the sheet, is

$$W > A, \quad W > 1, \tag{C 9}$$

and reduces when $V = 0$ to

$$|U_2 - U_1| > 2a, \tag{C 10}$$

the necessary condition for over-reflexion in the absence of a magnetic field. The other regimes are made possible only by the presence of a magnetic field. The second, for which (C 5) is satisfied on both sides, is

$$W < A, \quad W < 1, \quad W > A/(1+A^2)^{\frac{1}{2}}, \tag{C 11}$$

while the third, for which (C 4) is satisfied on one side but (C 5) holds on the other, is

$$W > \tfrac{1}{2}\left[A + \frac{A}{(1+A^2)^{\frac{1}{2}}}\right], \quad W > \tfrac{1}{2}\left[1 + \frac{A}{(1+A^2)^{\frac{1}{2}}}\right]. \tag{C 12}$$

Fejer (1964) has shown (taking three-dimensional disturbances into account) that the system is stable if

$$W < \tfrac{1}{2}[1 + A/(1+A^2)^{\frac{1}{2}}], \tag{C 13}$$

and in the incompressible limit $A \to \infty$ this criterion becomes $W < 1$, which is identical with (3.17). As the degree of compressibility increases, and A accordingly decreases, the right-hand side of (C 13) systematically decreases and compressibility exerts a destabilizing influence, with the criterion being $W < \frac{1}{2}$ in the limit of high compressibility ($A \to 0$). Equation (C 13) gives the lower stability boundary in figure 2. According to Duhau & Gratton (1973), however, once A drops below unity violation of (C 13) no longer automatically implies instability. Indeed, they find a second stable region when $A < 1$:

$$\frac{1}{2^{\frac{1}{2}}}\left[1 + \frac{A \times 2^{\frac{1}{2}}}{(1+A^2)^{\frac{1}{2}}}\right] < W < 1. \tag{C 14}$$

The fusion of criteria (C 9)–(C 14) leads to figure 2, in which two distinct stable over-reflecting regimes are evident.

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